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Geographic information — Features and Geometry — Part 2: Measure

Information géographique -
Caractéristiques géographiques et géométrie -
Partie 2 : Mesures

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i. Abstract

The documents in this series named "Features and Geometry" describes how geographic information is stored as data using a "Feature Model" are structured, accessed, and manipulated. The most important property is "location" which is represented as geometry in the coordinates of a CRS associated to a datum. The basis for the coordinates is a datum surface which is usually represented as an oblate ellipsoid. The most likely datum is the one is associated to GPS location systems which are derived from the ellipsoid WGS84 (https://gisgeography.com/wgs84-world-geodetic-system/). All numeric examples in this document use WGS84, but the technology is not specific, and the values for the semi-minor axis, semi-major axis and eccentricity can be adjusted for any reference ellipsoid.

This volume investigates accurate measurements of length and surface area on an ellipsoid. A metric is a function, system or standard of measurement. The curved nature of the ellipsoid does not support a simple "function-based metric" such as the Pythagorean metric on the plane nor the spherical metric on a sphere based on the central angle between two points.

On an ellipsoid, the curvatures of the surface changes with latitude (φ), and the nature of latitude and longitude (λ) differ also, so homogeneous functions such as used in the plane and the sphere do not work.

Spherical trigonometry is a rough approximation for the ellipsoid but in general do not take consideration scale which is dependent on latitude. Ellipsoidal geometry requires a Riemannian metric.

On an ellipsoid, the nature of the curvature at a point varies based on its relative position, and in the direction of the measurement. The Pythagorean metric on the plane, does work on the ellipsoid in small enough areas. The ellipsoidal metric takes (in a sense) both Pythagorean and spherical metric functions in small areas, and then calculate series of local measures and then combines them in summations (numeric integration).

ii. Keywords

The following are keywords to be used by search engines and document catalogues.

cartography	coordinate reference systems	curvature	radian
datum	differential geodesy	differential geometry	radius of curvature
ellipsoidal geometry	ellipsoidal metrics	first fundamental form	spherical geometry
geodesy	geographic database	geographic information systems (GIS)	location
geography	geometry	measure, measurement, metric	numeric integration

iii. Preface

This document "Features and their geometry: Part 2 Measure" deals with metrics for geometry associated to a curved surface, ellipsoid, that approximates the earth's surface. The theory of the mathematics dates to Isaac Newton (calculus, circa 1670), Gauss and Riemann (differential geometry, circa 1826-1855). It is doubtful that any of the procedures applied here are currently under patents.

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iv. Submitting organizations

The organizations and individual members of the Simple Features SWG (standards working group) reviewed and commented to produce this Document to the Open Geospatial Consortium (OGC).

The following are voting members of the OGC Simple Feature Standards Working Group**.**

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Features and Geometry – Part 2: Measure

1 Scope

This document describes how to measure length in meters and area in square meters of any curve or area on an ellipsoid (e.g. WGS84). Other ellipsoids may be used by adjusting the valued of the equatorial radius "a" and the polar radius "b" as appropriate. Ellipsoidal constants are kept in equations as variables; and can be used for any ellipsoid. Numeric examples use WGS84.

The only concept needed are numeric integration (4.13), radian (4.14) and radius of curvature (4.15).

The informative mathematics is in Annex A, and Annex B. The examples of the techniques for length and area are in Annex C.

1.1 Why this document is not a standard

This document describes the mathematics of the geometry described here as equations. This document informs anyone who implements these algorithms to measure geometries on the ellipsoid. The examples in the paper represented as tables were set up in spread sheets that would be simple loops in C++ or any programming language. Once a reasonable set of implementations are created; standards can then propose efficient interfaces to support the measured data.

1.2 Purpose

This document describes how measures of geometric objects on any ellipsoid (e.g. World Geodetic System 1984) are calculated, such as curves for length (in meters) and polygons for area (square meters). The WGS84 ellipsoid uses parameters in meters or ratios.

The measurements are not tied to a simplified representation as on a globe (a sphere) or on a map (a plane), but to the source of the data, e.g. the ellipsoidal coordinates of latitude (φ) and longitude (λ).

The equations and algorithms in this document work for any ellipsoid, but the numeric examples use WGS84 parameters. Angles (latitude and longitude) in calculations are in radians, because they easily convert angles in radians to arc distances in meters using the local radii of curvature Angles in examples, will normally be listed in both decimal degrees and radians, but calculations are in radians.

There are two type of radii that are required for measures.

- The radius of curvature of a **parallel** ($\rho(\varphi) = N(\varphi)\cos\varphi = a\cos\varphi\left(1-e^2\sin^2\varphi\right)^{-0.5}$) is in the plane of the parallel. The curvature is defined by the distance from the parallel to the polar axis. Because a parallel is a circle, the radius is constant along the parallel associated to its latitude, and only changes as the latitude changes. The radius at the equator is "a". The radius goes to 0 as the parallel nears the poles. See Figure 1 and Table 1.
- The radius of curvature of a meridian ($M(\varphi) = a(1-e^2)(1-e^2\sin^2\varphi)^{-1.5}$) is a function of latitude and varies from the equator (where the equatorial radius is "a") and decreases as the meridian approaches the pole (where the polar radius is "b"). The length of an arc (fixed as a angle) of latitude is longer in meters as the meridian approaches the poles. See Table 2.

The original definition of the Riemannian metric calculated these radii but used a different but valid method of calculating the radii of curvature. The equations in a Riemannian metric (A.3) are more complex than the radius of curvature (4.1 and 4.15). The Riemannian calculations are more complex but numerically identical. Figure 1 gives a direct geometric picture of the curvature along a parallel, $\rho(\varphi) = N(\varphi)\cos\varphi$.

If points on a map are input, then the inverse projection from the map (x,y) to the ellipsoid's latitude, longitude (φ,λ) surface should be used [18]. Small map areas, as in an engineering drawing, where x and y are really a local Euclidean survey and probably not projected from (φ, λ), then the usual Pythagorean metric suffices, and it does not require the formulae or methods here. The global metric needs the Pythagorean metric to be accurate "in small areas" (square kilometer, hectare) so that the summations (numeric integrals) presented below make that assumption. The local calculation to be "near-Euclidean" which require values of Δφ and Δλ be less than a degree to retain reasonable locally flat surface.

In addition to the 2D surface metric there is a 3D metric for a "near earth" use including latitude, longitude and elevation (φ, λ, h) with respect to the ellipsoid (mean-sea level). This "near surface" has an advantage over the ECEF (X,Y,Z) which would have to have the ability to keep curves directly associated to the "(*φ, λ, h*)" latitude, longitude and elevation point references involved, see [3], [4] and [19]. The algorithms for "(*φ, λ, h*)" extend "(*φ, λ*)" by adding elevation above the ellipsoid.

The numeric examples in this document will always follow the ellipsoidal surface (h=0). The extensions to the 2.5 D metrics in general use simple extensions to the 2D examples.

The circumference of a circle of radius "r" is "2πr", any arc along a circle whose length equals the radius "r" is called a radian (approximately 57.295779513…°). The arc length of a 1 radian angle is "r" and the arc length of 1° is r/57.295779513 or 0.017453292519968 r. A radian is a ratio of arc length to arc length and represented in equations as unitless measures (a ratio).

1.3 The importance of numeric integration

The formulae for calculating distances and areas on an ellipsoid would seem to require some integral calculus (see Annex B). The problem is that almost all the integrals do not have simple solutions. For this reason, in an application environment where the length or area of a feature is required to support some accuracy (we try for centimeter level accuracy in our examples). Annex B shows some simple numeric approximations to the needed values. Isaac Newton (1643-1727) or Leibniz (1646-1716) use a similar approximation technique to what he called the integral of a function:

Eq 1.

$$a = x_0 < x_1 < \ldots < x_n = b; \quad \Delta x_i = x_i - x_{i-1}$$

$$\int_a^b f(x)\,dx = \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \sum_{i=1}^{n} \left(\frac{f(x_i) + f(x_{i-1})}{2} \right) \Delta x_i \quad \text{*** Trapdezoid Rule}$$

$$\int_a^b f(x)\,dx = \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(x_i)\Delta x_i \quad \text{*** Newton's definition}$$

The first technique represented above for the formulae above is called the "trapezoid rule" or "trapezium rule" for numeric integration which converges faster than the second approach "rectangular" which is the original form from Newton’s definition. It is the purpose of this document is to use numeric integration while preserving accuracy in the answers while keeping the programs (a loop for the summation) simple and sufficiently accurate for GIS use, with the value of the area under the function approximated with smaller and smaller polygonal sides.

Another technique called "Simpson's Rule" can be more accurate for a set of (Δx_i), uses a parabolic approximation. In both techniques, as the maximum $\Delta x_i = |x_i - x_{i-1}|$ tends to zero, the numeric approximation to the integral $\int_a^b f(x)\,dx = \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(x_i)\Delta x_i$. See Table 2

The integrals we use comes from Gauss (1775-1855) and Reimann (1826-1866) in the 1850's to use integrals on ellipsoidal surfaces for geometry lengths and area (see Annex B). The key to accuracy of these summation loops is knowing how Δφ and Δλ measures are locally scaled to meters, by use of angles in radians and the appropriate radii of curvatures for meridians and parallels.

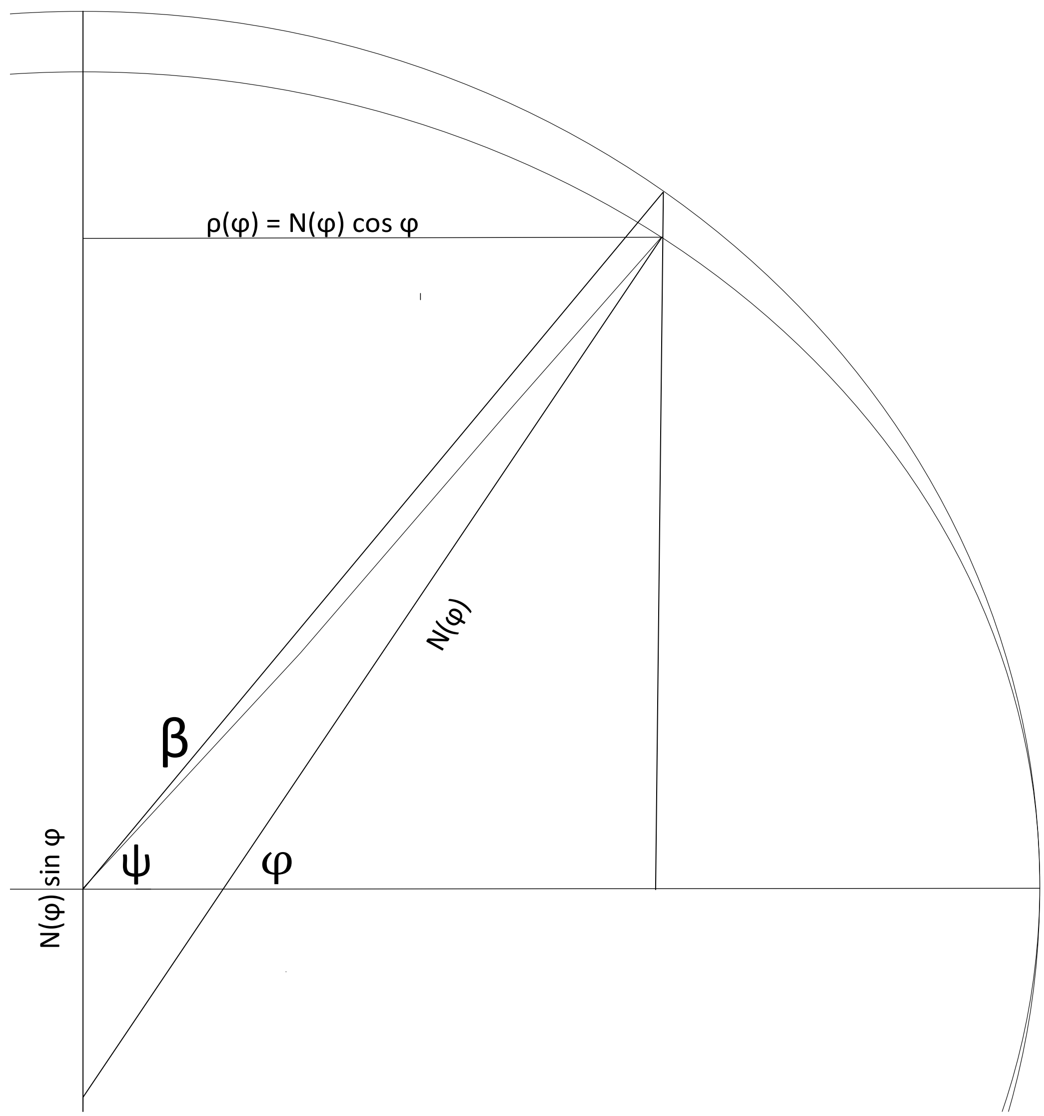


Figure 1 – Reduced latitude "β", geocentric latitude "ψ", and geodetic latitude "φ".

(larger eccentricity for emphasis)

1.4 The importance of the radius of curvature for calculating arc length

Each point "α" on a curve has a radius of curvature "r_α" which defines the best fitting circle tangent to the curve at that point. Along that fitted circle, the arc length of an arc segment of a small local angle "Δα" (expressed in radians) starting at the point has a length along the full circle of "$2\pi r_\alpha$" and a local distance along the curve of "$\Delta\alpha r_\alpha$". The "small arc" length of the arc and the length of the curve are very close. So much so that if the arc is "circle-like" such as a parallel (a circle of constant latitude φ, good for the entire curve) or an ellipsoid meridian (an ellipsoid, very near a circle) which works quite accurately for an angle of less than a degree.

Eq 2. Length of the normal: $N(\varphi) = \dfrac{a}{\sqrt{1-e^2\sin^2\varphi}}$

Eq 3. Radius of the parallel: $\rho(\varphi) = N(\varphi)\cos\varphi = \dfrac{a\cos\varphi}{\sqrt{1-e^2\sin^2\varphi}}$

Eq 4. Radius of the meridian: $M(\varphi) = \dfrac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}}$

In the figure below, the radius of the parallel is "*N(φ)cos φ*", and the length of the line that defines latitude "φ" from the surface of the ellipsoid to the polar axis is "*N(φ)*". The general equation for such a radius of curvature is in definition 4.1. Clynch [9] has a full description of these radii.

Eq 5. Meridian Distance $= \int_{\varphi}^{\varphi'} M(\varphi)\, d\varphi$

Eq 6. Parallel Distance $= \int_{\lambda}^{\lambda'} \rho(\varphi)\, d\lambda = \rho(\varphi)\, \Delta\lambda$ (if φ is constant, e.g. along a parallel)

Eq 7. Length of a curve:

$$f(t) = (\varphi(t), \lambda(t)) \Rightarrow$$
$$dist\left(p = f(t), p = f(t')\right) =$$
$$\int_t^{t'} \sqrt{\left(M(\varphi(t))\cdot\varphi(t)\right)^2 + \left(\rho(\varphi(t))\cdot\lambda(t)\right)^2}\, dt$$

Eq 8. Area in a polygon: $Area(A) = \iint_A M(\varphi)\rho(\varphi)\; d\varphi d\lambda$

The formulae for radii above are all that is needed to measure the length of short segments of arc of latitude and longitudes on the ellipsoid. The examples later imply use of arcs of length in of degrees 0.10° to 0.25°.

The integrals in Eq 8 and Eq 9 are difficult to solve in any closed form, and so the best was to calculate in a computer is the use of numeric integration as shown below in Eq 9 and Eq 10.

Eq 9. Length of a curve:

$$
\begin{aligned}
& dist(c(t):[t_0,t_n]\rightarrow(\varphi,\lambda);\\
& \Delta\varphi_i=\varphi_i-\varphi_{i-1};\quad \Delta\lambda_i=\lambda_i-\lambda_{i-1};\\
& [t_0,t_1,\ldots,t_n]=\left[(\varphi_0,\lambda_0),\ldots,(\varphi_n,\lambda_n)\right]\\
& dist(p,p')\cong\sum_{i=1}^{n}\sqrt{\left(\left(\frac{M(\varphi_i)+M(\varphi_{i-1})}{2}\right)\Delta\varphi_i\right)^2+\left(\left(\frac{\rho(\varphi_i)+\rho(\varphi_{i-1})}{2}\right)\Delta\lambda_i\right)^2}
\end{aligned}
$$

Eq 10. Area of polygon:

$$
\begin{aligned}
& \varphi_{i=0\ldots c}=\{\varphi_0,\ldots,\varphi_c\};\ \Delta\varphi_i=\varphi_i-\varphi_{i-1};\ i=1,\ldots,c;\\
& \lambda_{j=0\ldots r}=\{\lambda_0,\ldots,\lambda_r\};\ \Delta\lambda_j=\lambda_j-\lambda_{j-1};\ j=1,\ldots,r\\
& \Delta\lambda=\sum_{\substack{\Delta\lambda_i\\ \text{in area}}}\Delta\lambda_i\\
& A\cong\sum_{i=1}^{8}\left(\frac{M(\varphi_i)+M(\varphi_{i-1})}{2}\Delta\varphi_i\right)\left[\sum_{j=1}^{8}\left(\frac{\rho(\varphi_j)+\rho(\varphi_{j-1})}{2}\Delta\lambda_j\right)\right]\\
& \cong\left[\sum_{i=1}^{8}\left(\frac{M(\varphi_i)+M(\varphi_{i-1})}{2}\Delta\varphi_i\right)\left(\frac{\rho(\varphi_i)+\rho(\varphi_{i-1})}{2}\Delta\lambda\right)\right]\\
& \left(\frac{M(\varphi_i)+M(\varphi_{i-1})}{2}\Delta\varphi_i\right)\text{Height of strip, }\Delta\varphi_i\text{ in radians}\\
& \left(\frac{\rho(\varphi_i)+\rho(\varphi_{i-1})}{2}\Delta\lambda_j\right)\text{ Length of strip, }\Delta\lambda_j\text{ in radians}
\end{aligned}
$$

The final summation works best for $\Delta\varphi$ and $\Delta\lambda$ shorter that 0.25° (0.004363323 radians). The calculation in tables 1, 2, 3, and 4, result in errors of less than a millimeter.

1.5 The importance of the radius of curvature for calculating lengths and areas

The length of coordinate axes in the ellipsoidal coordinates (e.g. latitude (φ), and longitude (λ) are measured as angles, such as degrees, or radians, are not directly connected to the length of the arcs involved in meters. This requires us to convert angular arcs to the length in meters. The key is that an angle in radians is a function of the local radius of curvature. On a circle, it would be perfect, but meridians are ellipsoids, and the radius of curvature changes slowly getting longer as the latitude approaches the equator. This is the general idea for the use of Riemannian metrics. Riemannian metrics can produce exact measures by using integrals. Since there are not closed forms for these metrics, it is easier to use numeric approximation which derives from the original definitions for integrals stated by Newton. Taking significantly small arcs and applying the "trapezoid rule for numeric integration", gives us highly accurate and easily programmed numeric integration. This works because locally the Earth is "nearly flat", and the radii of curvature changes locally very closely, e.g. 0.25° arcs are nearly "a linear" rate of change in the local radius of curvature, both for latitude and longitudes. Which means the "trapezoid rule for numeric integration" is highly accurate in this sort of small areas. Examples in the annexes show that our application works for long arcs, one case is 0°to 90° gives us accuracy on the general error budget of a millimeter or less. Once the length algorithms, the area ones are similarly accurate.

1.6 Geometry and describing the position of features

The positions of features are represented as geometry, stored mostly as collections of algebraic curves, either as points, curvilinear features or boundaries of areas because these are things a computer can process. All of the integrals in this paper and in differential geometry in general, result in loops that aggregates the lengths of short curves or areas of small polygons that are then combined to create very accurate length or area of features, by using the scale factors in the information in Annex A and Annex B in the integrals in B.1. These integrals mimic classical Euclidean geometry, with the understanding that even for a curved surface, an area almost infinitesimally small works as Euclid and Pythagoras thought they would and change only with a near infinitely large numbers of infinitesimals are summed. It works when we get to Δφ and Δλ values less that a degree or 0.0174532925 radians e.g. π/180 radians, a radian is (180/π)°=57.29577951308232° or 57° 17' 44.80624").

These integrals are not directly "solvable" using integral calculus, so we back-up to the "summation" approximations of numeric integration. Within a 1°φ by 1°λ square the scale for both latitude and longitude vary slowly, see Table 1 and Table 2. In general, a 1°φ by 1°λ square can only nearly be considered "planar", especially away from the poles.

In all the numeric calculations in this document, the requirements of the mathematics are geodetic coordinates, (φ,λ), expressed in radians as required for the definitions, for the $\Delta\varphi$ and $\Delta\lambda$, which use the local radius of curvature (in meters) $M(\varphi)$ (along a meridian treated as a curve) and $\rho(\varphi)$ along a parallel treated as a curve. These can be used to scale angles to meters: $M(\varphi)\Delta\varphi$ along meridians and $\rho(\varphi)\Delta\lambda$ along parallels.

Note that both scaling factors depend solely on latitude(φ). These factors derive from the concept of the radius of curvature discussed in clause 1.4 below, which is the basis for spherical metrics, but can be used in small areas on the ellipsoid by using the local radii. First, think locally (using spherical metric based on the radii of curvature for the φ and λ axes (parallels and meridians). The elements of the Riemannian metric in Annex A and Annex B calculate local radii of curvature as defined in 4.1 and 4.15, but use different methods that approach them directly. Although the two approaches derive different formulae which are identical in value but different in form.

These curves and areas inherit their properties from both the coordinate space (an ellipsoid) from which they are collected and the underlying geometry of that space. In these processes, the implementors generally deal with one or more of 3 coordinate systems and their underlying spaces.

- E^3 (ECEF Earth-Centered Earth Fix Cartesian, (X, Y, Z) or (X_i, i=1, 2, 3) ⇄
- S^2: (spheroid, ellipsoid, sphere), geodetic (φ, λ) or geocentric (ψ, λ) ⇄
- E^2: map (x, y) or $(x_i, i=1,2)$

Of these systems, only the first (which is used in GPS systems) uses "standard Euclidean geometry" which implies a Pythagorean metric $\left(d = \sqrt{\sum \Delta x_i^{\,2}}\right)$, the square root of the sum of the delta-coordinates squared. The last two systems have metrics that differ in form from the Cartesian distance based on Pythagoras. There is no universal "equation-based" metric spheroidal distance or map distances with the corresponding scale, which represent real distances until the projection is mapped back to the spheroid. For example, a north-south circumnavigation along a meridian is *40,007.862 km* (see Table 2), and an east-west circumnavigation along the equator is 40,075.016 km, (see Table 1)

Spherical trigonometry works well on a "sphere", but even the small eccentricity of reference ellipsoids will alter both the distance and area measurements. In the example above of circumnavigations can differs by about 70 kilometers. To a lesser extent, a degree of latitude will vary slightly along a meridian because of the eccentricity and because the differences in curvatures (both along the parallel and the meridians) is dependent on latitude. Because of the same flattening, the degree of latitude also varies but only slightly (1.121 km); 110.574 km at the equator and 111.694 km at the pole for a distance difference between the equator and the pole of only about 1%.

Some maps can often use the standard Pythagorean metric for relatively small areas, such as engineering drawings over relatively small areas. The smaller area maps can get away with it because at that size the micro-geometry works as Euclid visualized it. These engineering drawings are not projection and therefore not really a topic for this paper.

Metrics on curved surfaces (such as an ellipsoid) embedded in $\mathbb{E}^3$ inherit a Riemannian metric from that embedding. Distance on a curved surface between two points P_1 and P_2 is the length of the shortest curve (the geodesic) on the surface that begins at P_1 and ends at P_2. If the surface is a plane, the derived metric is Euclidean. However, if the surface is curved, such as the ellipsoid, to get an accurate distance along the geodesic there are two options, both involving the curves that link the two points. The length of the shortest curve on the surface is the "distance" between the points. Once this curve is identified, the length of the curve is an integral, either in $\mathbb{E}^3$(X, Y, Z) or on the spheroid $\mathbf{S}^2$(φ, λ). As long as the representation of the $\mathbb{E}^3$ version of the curve fits the S^2 version of that same curve, the result will be the same (possibly with slightly different round-off errors).

The Earth-Centered Earth-Fixed Cartesian coordinate space ($\mathbb{E}^3$) places the center of the ellipsoid or spheroid at the origin (X, Y, Z)=(0, 0, 0), where the Greenwich Meridian (λ=0; longitude) as a curve on the spheroid that passes through the positive X-axis and is contained in the half plane where Y=0. The Y-axis plane is 90° East and the Z-axis is the rotational axis, and the positive Z+-axis passes through the north pole.

This means the spheroid $\mathbf{S}^2$ is a topological sphere, with its center at the $\mathbb{E}^3$ origin, rotating about the Z-axis. Some systems ignore the eccentricity and use a sphere (mimicking a globe). This eccentricity was first hypothesized by Isaac Newton and was verified by measuring the distances between latitudes that would be equal on a sphere but different on an ellipsoid, further near the poles than near the equator. A grade measurement expedition (1735-1738) by the French Academy of Science to Lapland and Peru verified the oblateness. Later efforts by F. G. W. von Struve and Bessel in 1814 were used for the Bessel Ellipsoid (1841) with an inverse flattening of 299.1528 (WGS 1984 ellipsoid uses 298.2572).

A linear curve called a "line" taking it name from the formulae that in Euclidean (Cartesian) spaces is Euclid's line. But the algebra of the "line" segment between points $\vec{p} = (\varphi_p, \lambda_p)$and $\vec{q} = (\varphi_q, \lambda_q)$ on a spheroid is simply a "linear" equation, using vector forms of $\vec{p}$ and $\vec{q}$, $\vec{c}(t)$:[0,1] → S^2 :[$\vec{c}(t) = (1-t)\vec{p} + t\vec{q}$]; the underlying surface gives lines a curvature. The geometry follows the surface of the coordinate space. The properties of the curve depend on both its algebra, and the way the coordinates are used to be associated to the Earth's surface. Flat maps tend to be used as if they were Cartesian (as defined by René Descartes (1596-1650)) and therefore aligned to Euclidean geometry. However, globes aren't Euclidean, they are either spheres modeled by spherical trigonometry or ellipsoids which are close to sphere, but not easily modeled until Gauss (1777-1855) and Riemann (1826-1866) used

Newton's (1642-1726) or Leibniz's (1646-1716) calculus to define "metrics" for other surfaces, not by equations but by integrals.

This standard enumerates the various mechanisms for representing feature positions as geometry on a curved surface spatially embedded in $\mathbb{E}^3$ or derived from such surfaces (e.g. map projections). The core difference between these geometries is the calculation of distances and associated lengths. Classical computer programs using coordinates work in Cartesian (and thereby Euclidean) coordinate spaces $\mathbb{E}^2$ and $\mathbb{E}^3$ which use a Pythagorean metric. Modern geodesy does its calculation generally in one of two manners: intrinsic or extrinsic metrics.

The intrinsic methods are based on operations on the surface being used, which usually involve differential geometry developed by Gauss and Riemann (most important the vector product) for curved space, usually creating integrals and not formulae for the calculation of lengths and areas (see Bomford [3] and Zund [38]). An example of a purely intrinsic method are the multiple measures of lengths of degrees of latitude and longitude which implied first spherical models and later with more data and more accurate data implied a spheroidal model (oblate spheroid).

The extrinsic methods take measures in a larger space and then interpret information about a surface embedded in this larger space (S^3). The best example is the GNSS satellite systems which interpret transmission time to multiple distance measures to calculate a position on the ellipsoid. In this later approach, the integrals can use the $\mathbb{E}^3$ coordinates (X, Y, Z) and transform to geodetic coordinates (φ, λ) i.e. latitude, longitude.

1.7 The importance of geodesy

Geodesy is the science of the shape and gravity of the Earth, specifically for the geospatial community, this has implications for the types of geometry embedded in geographic coordinate reference systems based on geographic datums and their reference ellipsoid. These coordinates expressed as positions on the ellipsoid, geoid or surface of the Earth involve positions defined with respect to the equator[1], and the Prime Meridian (Greenwich) and "local vertical" offsets from that surface (elevation or depth). Once the coordinate system is defined, the geometry is expressed in that coordinate system, see ISO 19111 for the coordinate reference systems (CRS) and ISO 19107 for representing geometry in any coordinate systems. For example, the Euclidean line with all its properties cannot exist on a curved closed surface like a spheroid because the properties of such a surface prevent the fundamental infinities embedded in the concept of a line. A curve may seem to be a line in a small area, but the infinity of its length in Euclid's geometry cannot exist in a closed finite surface like a globe or an ellipsoid.

If a geographic representation system uses Euclidean geometry in 2D ($\mathbb{E}^2$ such as extrapolating from maps) or 3D ($\mathbb{E}^3$ either addition of elevation to an $\mathbb{E}^2$ system or embedding a 2D "Earth-like" (geoid) surface in an earth centered earth fixed coordinate system, ECEF), then they are engineering coordinate reference system ($\mathbb{E}^2$ or $\mathbb{E}^3$)and should not to be confused with a projection where the

[1] The nominal measure of position with respect to these lines are expressed in angles, for latitude measured by the angular direction of the local surface vertical as it crosses the equatorial plane. For longitude this is the central angle for the reference ellipsoid, as a rotation parallel to the equatorial plane with respect to the prime meridian. If the ellipsoid is a not a sphere the latitude "φ" is not the same as the central angle "ψ".

target coordinate space does not represent a surface with Cartesian, Euclidean and Pythagorean functions.

Because the Earth is not flat, the geometry of a geoid surface is non-Euclidean and all formulae on the plane that depend on Euclidean geometry, such as the Pythagorean Theorem, are usually invalid. There are several mechanisms that can be used to work around this, in differing order of functionality and software performance.

1.8 The importance of geometry on the ellipsoid

By our own common experience, if we are close enough to the earth's surface, Pythagorean metrics work. This is because that "close enough" looks like and works like a flat plane. So locally, the geometry is what we already know how it works. What differential geometry does is to "integrate" these small pieces into continuous realization of aggregating all the short parts with accurate where we add them to realize the length or areas, by aggregating all the little parts.

This leads to two problems: how do we convert latitude "φ" and longitude "λ" to meters (or feet). At the equator, we have almost 25,000 miles of longitude and at the poles we can stand on all the longitudes at 89° 59' 59.999" (N) latitude, is about 30cm. The "parallel" radius of curvature the earth at 90°-.001"= 89° 59' 59.999" is r=6,356,752.314245m; changing .001" converted to radians, multiplied by r to convert to meters is 30cm. Geometry gives you the scales (local radius of curvature) that allow you to measure angles (latitude along meridians and longitude along parallels) in radians to meters $r_\varphi \Delta\varphi$ and $r_\lambda \Delta\lambda$, in the direction of latitude or longitude; see clause 1.4.

2 Conformance Classes

A feature is a representation of a real-world object. For spatial applications, the most important properties of a feature are its location and shape. The most technically difficult part of geospatial application is dealing with the geometry that represents that location, usually on a map, ellipsoid or geoid. In small areas, Euclidean geometry work fine. However, the larger the area the greater the need to compensate for the Earth's curvature which can be dealt only with a non-Euclidean geometry engine.

The conformance classes in this document depend on the methods used for operations for geometry objects used for the spatial extents of features.

3D ECEF: 3D Earth Centered Earth Fixed: Use an appropriate ellipsoid and convert all coordinates to $\mathbb{E}^3$ and use integration and differential equations to make calculations in $\mathbb{E}^3$. See Burkholder [4]. An ECEF coordinates system is a right-handed X, Y, Z coordinate system. The geometry of features must be contained on the reference ellipsoid.

Ellipsoidal Geometry: Use an oblate ellipsoid having a fixed equatorial radius "r_e" or "a", a slightly smaller polar radius "r_p" or "b" and use an ellipsoidal metric from differential geometry consistent with the geoid's radii. See Ligas, Panou [27], Eisenhart [13], Hotine [16], Lund [24], Struik [29], Zund [38], and more specifically differential geometry.

3 References

The following normative publications in their most recent form contain information important to this document.

OGC-17-087r10 Geographic information — Features and geometry – Part 1: Feature models
ISO 19101-1 Geographic information — Reference model
ISO 19103 Geographic information — Conceptual schema language
ISO 19107 Geographic information — Spatial schema
ISO 19108 Geographic information — Temporal Schema
ISO 19111 Geographic information — Spatial referencing by coordinates
ISO 19126: Geographic information — Feature concept dictionaries and registers
ISO/IEC 13249-3 - Information technology — SQL Multimedia and Application Packages - Part 3: Spatial
ISO 19162: Geographic information — Well-known text for coordinate reference systems

4 Definitions

In addition to the list below, any definition in any normative reference will be acceptable. All Standard English words are in either in the Oxford or Webster's dictionary, usually both, sometimes with variants in spelling. The better dictionaries of the English tend to list all allowable alternate spellings based on national usages and custom.

4.1 curvature (of a curve)

<differential geometry> second derivative of a curve parameterized by arc length, κ, at a point

Note to term: The curvature is the reciprocal of the radius of curvature (4.15)

Note to term: The radius of curvature of an arc is the radius of the best fitting circle of the curve at that point.

Eq 11. curvature of curve "c": $\kappa = \dfrac{c''}{\left(1+\left(c'\right)^2\right)^{3/2}}$

See: Concise Dictionary of Mathematics, "curvature" [5]

4.2 ellipsoid
reference ellipsoid

<geodesy> geometric reference surface represented by an ellipsoid of revolution, that is a surface of rotation around the polar axis so that the equatorial radii are all equal

4.3 ellipsoidal (geodetic) coordinate system

<geodesy> coordinate system in which position is specified by geodetic latitude, longitude and (in the three-dimensional case) ellipsoidal height

Note to term: Geodetic latitude is measured by the angular direction of the local normal to the equatorial plane. See Figure 1

[ISO 19111]

4.4 ellipsoidal geocentric coordinate system

<geodesy> coordinate system in which position is specified by geocentric latitude and longitude

Note to term: Geocentric latitude is measured by the direction of the line from the center (0, 0, 0) in (X, Y, Z) of the ellipsoid to the surface.

4.5 ellipsoidal height, "h"

<geodesy> distance above the reference ellipsoid along the local ellipsoidal principal normal

Krakiwsky [21], Clynch[8]

4.6 engineering coordinate reference system

coordinate reference system based on a local reference describing the relationship of points to a Euclidean coordinate system

Note to term: Any engineering coordinate system use a Euclidean metric, i.e. a subset of $\mathbb{E}n$ usually at most 3 spatial and, optionally, 1 temporal.

4.7 first fundamental form (in differential geometry)

inner, dot or vector product on the tangent space of a surface in three-dimensional Euclidean space $\mathbb{E}^3$ which is derived canonically from the dot product (inner product) on the tangent space of a surface derived from the "vector dot product" in three- dimensional Euclidean space ($\mathbb{E}^3$)

Note to term: The first fundamental form is a Riemannian metric tensor usually derived from the embedding of the surface in $\mathbb{E}^3$. See [1] for tensors. The same mechanism can be used if the embedding is replaced with an isometry (a mapping which preserves distance). It is not necessary to change the extent of the metric. For example, the first fundament form for 3D polar coordinates should be functionally equivalent to standard polar coordinates used in physics (ρ,θ,φ), where ρ is the distance from the origin, φ is rotation from the x-axis towards the y-axis, and θ is rotation from the x-y plane towards the z-axis.

Note to term: Alternatively, the first fundamental form can be derived by calculating the radius of curvature, (see 4.15)

See Annex B.

4.8 geodesy

scientific and technical discipline addressing the fundamental basis of positioning and localization of geographical information science that studies dimensions, shape and the gravity field of the Earth

Note to term: Both definitions above derived from IGN (translated from French). These definitions reflect both the purpose and the practice of geodesy. The purpose is to rationally locate positions on the earth which requires the practice of investigation of the shape and gravity of the planet (in this document, the Earth).

IGN [17]

4.9 geocentric latitude, ψ, φ', φ_c, and sometimes, ϕ or φ

⟨geodesy⟩ angle to the equatorial plane of the line from the center of the ellipsoid to the surface of the ellipsoid at the point referenced, positive north, negative south (in radians for the calculations in this document)

Note to term: Unlike geodetic latitude, the line that determines geocentric latitude passes through the geometric center of the ellipsoid but is not always perpendicular to the reference ellipsoid surface.

Krakiwsky [21]

4.10 geodetic latitude, φ, φg, and sometimes φ

⟨geodesy, astronomy⟩ angle that the normal at a point on the reference ellipsoid makes with the plane of the equator, positive north, negative south (in radians for the calculations in this document)

Note to term: The line that determines geodetic latitude is perpendicular to the reference ellipsoid and usually does not pass through the center of the ellipsoid, except along the equator or at the poles. The following are valid for the surface of the ellipsoid, where geodetic "$\varphi=\varphi_g$" and geocentric "$\psi=\varphi_c$" latitudes and "λ" longitude.

Eq 12. Geodetic latitude: $$\varphi = \arctan\left[\frac{a^2 \tan\psi}{b^2}\right] = \arctan\left[\left(1-e^2\right)^{-1} \tan\psi\right] \quad \text{i.e.} \quad \tan\varphi = \frac{a^2}{b^2}\tan\psi$$

Eq 13. Geodetic longitude: $$\psi = \arctan\left[\frac{b^2 \tan\varphi}{a^2}\right] = \arctan\left[\left(1-e^2\right)\tan\varphi\right] \quad \text{i.e.} \quad \tan\psi = \frac{b^2}{a^2}\tan\varphi$$

Note to term: If ψ= φ then they are 0°, or ±90°. The tangent at these angles has value of either 0 or ±∞.

Krakiwsky [21]

4.11 geoid

⟨geodesy⟩ equipotential reference surface of the Earth's gravity field which is everywhere perpendicular to the direction of gravity and which best fits a mean sea level either locally or globally

Note to term: Geoids are usually represented as differences from a reference See: Concise Dictionary of Mathematics, "curvature" [5]

ellipsoid.

4.12 metric, measure

function, system or set of algorithms that returns a numeric measure of some notional property such as distance or surface area, or any measured property possessed by an entity or set of entities

Note to term: In this document, the metric will speak to the measure of the length of curves or to the area of a surface. The basic units of measure in this paper will be the meter and square meter or aggreges of these such as kilometer or hectare (10,000 square meters).

4.13 numerical integration

numeric methods to approximate values for definite integrals

Note to term: It may be that there is no analytical method of finding an antiderivative of the integrand. Among the elementary methods of numerical integration are the trapezoid rule or Simpson's rule.

See: Concise Dictionary of Mathematics, "numeric integration" [5]

4.14 radian (rad)

<mathematics> measure of an angle base on a portion of a circle, a full circle being 2π radians,

Note to term: 1°=0.01745329252 radian= π/180°; 1 radian=180°/π=57.29577951°. So, an angle in degrees times π/180°is converted to the same angle in radians.

Note to term: All integrals in this standard use radians as a measure of angle. The most important issue is the use of numeric integration where $\Delta\varphi$ and $\Delta\lambda$ will be expressed in radians, not in degrees. If the curve for $\Delta\varphi$ has a local radius of curvature of "r_φ" and similarly for $\Delta\lambda$, and a radius of curvature of the local λ-axis is "r_λ" then the lengths of the arcs are approximately $r_\varphi \Delta\varphi = arclength(\Delta\varphi)$ and $r_\lambda \Delta\lambda = arclength(\Delta\lambda)$ dependent on the variations of the variance of the radius of curvature along the arcs of Δφ and Δλ. If Δφ and Δλ are small enough (at least smaller than a degree), then the total length is:

Note to term: Tables may use two columns one for each angle; in degree (°) and in radian (no unit of measure). Radians are considered a ratio, between a circular arc length and the corresponding radius of that circle. Since the "a" and "b" (the two axes length of the ellipsoid in meters) the result of any of the numeric integrals below will be in meters or squared meters.

Note to term: The radian appears in mathematical literature in 1871, but the concept derives from the middle ages, from Arabian mathematics.

4.15 radius of curvature

radius of the circle which best fits a curve at a point

Note to term: The radius of curvature is the reciprocal of the curvature (of a curve)4.1).

Eq 14. Radius of curvature: $\rho = 1/\kappa$ where $\kappa = \dfrac{c''}{\left(1+(c')^2\right)^{3/2}}$

Note to term: If Δφ and Δλ are both less than 0.0043633231 radian (.25 degree), the accuracy of the combined distances can have sub-meter or better accuracy (see radian (4.12) and Annex C). Zooming into smaller and smaller Δφ and Δλ will eventually produce better accuracy for the calculated arc length. The radius of curvature function along a parallel is $\rho(\varphi) = N(\varphi)\cos\varphi$ and along a meridian is $M(\varphi) = \dfrac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}}$, see Eq 9 for the integral Eq 5 above, and the numeric integration in Table 2 below. Using integral equations, the distance between longitudes in meters along the same latitude (φ), the length of the interval (λ_0, λ_1) is $\int_{\lambda_0}^{\lambda_1} \rho(\varphi) d\lambda$. The distance between latitudes along the same longitude (λ), the interval (φ_0, φ_1) is $\int_{\varphi_0}^{\varphi_1} M(\varphi) d\varphi$.

Note to term: If $\Delta\lambda_n = |\lambda_n - \lambda_{n-1}|$ were the two points are on the same parallel, i.e. are both the same "φ", then $\rho(\varphi_n)\,\Delta\lambda_n$ is the exact arc distance value in meters on the ellipsoid. If φ_n and φ_{n-1} are not equal, then a reasonable approximation between the two λ's is $\left[\frac{\rho(\varphi_n)+\rho(\varphi_{n-1})}{2}\right]\Delta\lambda_n$. The $M(\varphi)\Delta\varphi$ with a non-zero arc length for $\Delta\varphi_n = |\varphi_n - \varphi_{n-1}|$, then a reasonable approximation of the arc length for φ is $\left[\frac{M(\varphi_n)+M(\varphi_{n-1})}{2}\right]\Delta\varphi_n$. (see [32]).

Eq 15. Delta Distance: $\Delta d = \sqrt{\left(r_\varphi \Delta\varphi\right)^2 + \left(r_\lambda \Delta\lambda\right)^2}$

Eq 16. Radius of Meridian: $r_\varphi = M(\varphi) = \dfrac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}}$

Eq 17. Radius of Parallel: $r_\lambda = \rho(\varphi) = N(\varphi)\cos\varphi$

See: "curvature"

4.16 reduced latitude (of a point of latitude φ)
parametric latitude
β

angle from the reference ellipsoid center to the point directly above the equator on the same line parallel to the polar axis of the point of latitude φ on the reference ellipsoid to the point on the **surrounding sphere.**

Eq 18. Reduced latitude:

$$\begin{aligned}\beta(\varphi) &= \arctan(\sqrt{1-e^2}\,\tan\varphi) \\ &= \arctan((b/a)\tan\varphi) \\ &= \arctan\left((1-f)\tan\varphi\right)\end{aligned}$$

Note to term: Figure 1 shows the differences between both geodetic and geocentric latitude "φ, ψ" and the corresponding reduced latitude "β". The figure uses a larger eccentricity that would normally be seen on a reference ellipsoid

4.17 Riemannian metric $g(\vec{u},\vec{v}) = \vec{u}\cdot\vec{v} \in \mathbb{R}$

smooth function on a manifold M (e.g. surface) that defined a continuous inner product $g(\vec{u},\vec{v}) = \vec{u}\cdot\vec{v}$ (sometimes called the "dot product") at each point "$\boldsymbol{x}$" on the manifold on the tangent spaces $T_x(M)$ at each point on M

Note to term: On an ellipsoid, $d_\varphi \cdot d_\varphi = M^2(\varphi)$and $d_\lambda \cdot d_\lambda = (N(\varphi)\cos\varphi)^2$ and $d_\varphi \cdot d_\lambda = 0$. The values of M(ϕ) is the radius of curvature for the meridian at the latitude φ, and N(φ)cosφ is the radius of curvature for the parallel of latitude.

4.18 surrounding sphere (of the reference ellipsoid)

sphere centered at the origin of the reference ellipsoid with a radius equal to the ellipsoid's equatorial radius (semi-major axis "a").

5 Measure for an ellipsoidal (φ, λ) coordinate systems and geometry

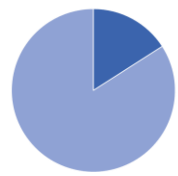
A radian has arclength "r" on a circle of radius "r". The circumference is 2 π r= π d. All angles are expressed in radians.

1 radian= 57°.295779513…= (180°/π)

Circle=2π= 6.283185307…

1 radian=1/6.283185307… of the circle

1 degree=π/180= .017453292519943…



The applications work with geometry in the standard geodetic coordinate system geodetic latitude (φ), longitude (λ), and ellipsoidal height, if needed, (φ, λ, h). The following example deal with two corners of a latitude-longitude rectangle, with sides of two meridians and two parallels with two corners (φ_o, λ_o) and (φ_1, λ_1) with NS and EW distances are generally less than a quarter degree.

All angle in the equation for φ, λ, Δφ and Δλ are used in calculations in radians. All distance expressions along curves in (ϕ, λ) are in meters.

The north-south distance between 2 points, (φ_0, λ_0) and (φ_1, λ_1) projected on the same meridian (φ) has a north-south distance of approximately:

Eq 19. NS distance: $dist_{n-s} = \left(\dfrac{M(\varphi_0)+M(\varphi_1)}{2}\right)\Delta\varphi = r_\varphi \Delta\varphi$ where $\Delta\varphi = |\varphi_1 - \varphi_0|$.

The east-west distance between 2 points, (φ_0, λ_0) and (φ_1, λ_1) projected on the parallel (λ) has an east-west distance of approximately:

Eq 20. EW distance: $dist_{e-w} = \left(\dfrac{\rho(\varphi_0)+\rho(\varphi_1)}{2}\right)\Delta\lambda = r_\lambda \Delta\lambda$ where $\Delta\lambda = |\lambda_1 - \lambda_0|$.

The distance between 2 points and the area of bounded rectangle $[(\varphi_0, \lambda_0), (\varphi_1, \lambda_1)]$ (

Eq 21. Combined distance: $dist = \sqrt{\left(r_{\varphi}\Delta\varphi\ \right)^{2} + \left(r_{\lambda}\Delta\lambda\ \right)^{2}}$; and $area = \left(r_{\varphi}\Delta\varphi\ \right)\left(r_{\lambda}\Delta\lambda\ \right)$

Are expressed in meters where $r_{\varphi} = \dfrac{M(\varphi_0) + M(\varphi_1)}{2}$ and $r_{\lambda} = \dfrac{\rho(\varphi_0) + \rho(\varphi_1)}{2}$. The length of a curve is calculated by Eq 104. The rectangle with the diagonal $[(\varphi_0,\lambda_0),(\varphi_1,\lambda_1)]$ has $area = \left(r_{\varphi}\Delta\varphi\right)\left(r_{\lambda}\Delta\lambda\right)$ in square meters. The area of a surface shall be calculated by equation Eq 96. The radii of curvature for latitude and longitude are functions of φ, where "a" is the equatorial radius and "e" is eccentricity; see 4.1.

Eq 22. Length of normal: $N(\varphi) = \dfrac{a}{\sqrt{1 - e^2 \sin^2 \varphi}}$

Eq 23. Radii of longitude: $\rho(\varphi) = N(\varphi)\cos\varphi = r_{\lambda}$

Eq 24. Radii of latitude: $M(\varphi) = \dfrac{a(1-e^2)}{(1 - e^2 \sin^2 \varphi)^{3/2}} = r_{\varphi}$

These radii can be taken to 14 significant digits, based on the lengths of the two radii of the ellipsoid, for the equatorial axis and polar axis. The flattening is also an algebraic function of the two ellipsoidal radii.

Eq 25. Ellipsoid radii:

$a = 6,378,137.0$ m equator by definition

$b = 6,356,752.314245180$ m polar by calculation $b = a(1-f)$

The radii of curvature here depend on the two radii of the ellipsoid, which are defined by "exact values" means if the functions are taken as double precisions valid digits, the values of the above functions can be as many as 14 to 17 decimals digits. This class uses an $\mathbb{E}^3$, an Earth Centered Earth Fixed coordinate system where the reference ellipsoid is centered at the (0,0,0) with the Z-axis containing the polar access, and the X-Y plane containing the equator, and the intersection with the Greenwich "0° meridian" and the equator is on the X and Y-axis. Theses following lay out how geometry should be done on a reference ellipsoid, usually embedded in $\mathbb{E}^3$, in general the types of analytic surfaces use within a Datum.

Eq 26. Point: $point = \{(\varphi,\lambda)\}$

Eq 27. Curve: $curve = \{p_i = (\varphi_i,\lambda_i)\} = \{p_0 = (\varphi_0,\lambda_0), p_1 = (\varphi_1,\lambda_1), p_2 = (\varphi_2,\lambda_2), \ldots, p_n = (\varphi_n,\lambda_n)\}$

$$surface = \left[p_{i,j} = \left(\varphi_{i,j}, \lambda_{i,j} \right) \right]_{0,0}^{m,n}$$

Eq 28. Surface:

$$= \begin{bmatrix} p_{0,0} & p_{0,1} & p_{0,2} & p_{0,3} & \cdots & p_{0,n-1} & p_{0,n} \\ p_{1,0} & p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n-1} & p_{1,n} \\ p_{2,0} & p_{2,1} & p_{2,2} & p_{2,3} & \cdots & p_{2,n-1} & p_{2,n} \\ p_{3,0} & p_{3,1} & p_{3,2} & p_{3,3} & \cdots & p_{3,n-1} & p_{3,n} \\ p_{4,0} & p_{4,1} & p_{4,2} & p_{4,3} & \cdots & p_{4,n-1} & p_{4,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{m-3,0} & p_{m-3,1} & p_{m-3,2} & p_{m-3,3} & \cdots & p_{m-3,n-1} & p_{m-3,n} \\ p_{m-2,0} & p_{m-2,1} & p_{m-2,2} & p_{m-2,3} & \cdots & p_{m-2,n-1} & p_{m-2,n} \\ p_{m-1,0} & p_{m-1,1} & p_{m-1,2} & p_{m-1,3} & \cdots & p_{m-1,n-1} & p_{m-1,n} \\ p_{m,0} & p_{m,1} & p_{m,2} & p_{m,,3} & \cdots & p_{m,n-1} & p_{m,n} \end{bmatrix}$$

The simplest geometry figure is a point, a set of one position. Since within a point, there can be no motion without leaving, the point has zero-degrees of freedom (of motion) and is therefore referred to as a 0-dimensional geometry. The next geometry up the scale of dimension is a curve. On a curve, motion along the curve has one degree of freedom and is therefore a 1-dimensional geometry figure. Following this pattern, surfaces are 2-dimensional, and solids are 3 dimensional. All geometry objects are sets of position and so every implementation of a geometry must have a manner to represent positions.

Measuring geometry, the array of *(φ, λ)* pairs should coincide with the dimension of the geometry. In dealing with a point, only one control point is involved. In dealing with a curve, the sample points will be a sequential array of sample control points,

A geometry representation shall use a single coordinate reference system (CRS) to express position(s) in space. The CRS shall be consistent throughout any primitive object (point, curve, surface or solid). The geometry objects or datatypes shall contain an identifier for the CRS in use or inherit one from a container.

If the datum is dynamic, the CRS reference shall contain the epoch for which the coordinates are valid. The authority for a dynamic datum should be considered as the primary source of information concerning adjustment between epochs of the datum.

The application should be able to calculate the length and areas of feature geometries in the ellipsoidal coordinate system (φ, λ) or (φ, λ, h) based on integration either in $\mathbb{E}^3$ or on the ellipsoid using a first fundamental form for that ellipsoid. If the reference ellipsoid is a sphere, the application can use spherical trigonometry to calculate distances without using integration techniques. Using a sphere is inconsistent with the actual geometry, which is an ellipsoidal. Any geometric representation of feature position requires an understanding of the "rules" of the space where this geometry is created.

6 Geometry on a curved datum reference surface

This set of requirements lays out the requirements of doing geometry on a curved surface embedded in $\mathbb{E}^3$, in general the types of analytic surfaces use within a Datum, e.g. the reference surface or ellipsoid. See Burkholder [4].

The issue is that in E^3 geometries are in coordinates in (x, y, z) with the restriction stated in the above

Measurements of distance or length, area or volume shall be equal with those that can be calculated on the reference ellipsoid using a Riemannian Metric (first fundamental form) (see 4.17) with a stated error budget. The values in the measurements are the local radii of curvature on the meridian and the parallel.

Note: The Riemannian metrics described Annex A works directly with geodetic coordinates, (φ, λ); with geocentric coordinates (ψ, λ). The geocentric system (ψ, λ) is simpler, but the geodetic system (φ, λ) is more commonly used. Both systems are based on local radii of curvature (see Table 1 and 2 in Annex C).

The simpler geocentric system may be use in place of the more complex geodetic system by transforming φ to ψ and back as necessary, λ remains the same in these transformations.

Geometry dependent on these systems such as geodesics (shortest distance), rhumb lines (constant bearing) and any geodesic circle (constant distance from a center point), shall be consistent with calculation in the Earth centered, Earth Fixed $\mathbb{E}^3$ coordinate system in which the datum surface is defined, with a stated error budget, and shall be consistent with the same geometries using ellipsoidal calculations in latitude (φ) and longitude (λ).

7 Geometry in map projection spaces with datum information

"It is possible to derive a set of formulae to convert geographic coordinates to grid coordinates in purely mathematical terms. In general, equations can be derived of the form. see [18] and [28].

Eq 29. Map position for (φ,λ): $(E, N) = f(\varphi, \lambda)$

Eq 30. $$Length(\varphi_s, \varphi_e) = \int_{\varphi_s}^{\varphi_e} M(\varphi) d\varphi$$

Eq 31. Lenght of a Merridian between φ_s to φ_e: $$\varphi_s, \ldots, \varphi_e = \varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_n; \Delta\varphi_i = |\varphi_i - \varphi_{i-1}|$$ $$d_m(\varphi_s, \varphi_e) \cong \sum_{i=1}^{n} \left(\frac{M(\varphi_{i-1}) + M(\varphi_i)}{2} \right) \Delta\varphi_i$$

Eq 32. Length of a parallel from λ_s to λ_e: $\Delta\lambda = |\lambda_e - \lambda_s| \Rightarrow d(\lambda_s, \lambda_e) \cong \rho(\varphi)\Delta\lambda$

In other words, the position in a map space (x, y) = (E, N) → (φ,λ) is mapped to a position on the reference ellipsoid. The argument also implies a display or digital map should be associated to the map projection and which, a map or map-like display on a screen should be able to map back to geodetic coordinates

8 Numeric Integrals for Ellipsoidal Measures

8.1 Length of a Meridian Segment

Along a meridian, the radius of curvature is dependent on φ. See Table 2.

Eq 33. Exact Integral: $$Length(\varphi_0,\varphi_n)=\int_{\varphi_0}^{\varphi_n} M(\varphi)d\varphi$$

$$\varphi_0<\varphi_1<\varphi_2<\ldots.<\varphi_n$$

Eq 34. Numeric Integral: $$Length(\varphi_0,\varphi_n)\cong\sum_{i=1}^{n}\left(\frac{M(\varphi_i)+M(\varphi_{i-1})}{2}\right)\Delta\varphi_i$$

8.2 Length of a Parallel Segment

The radius of curvature for a parallel is dependent on the latitude of the parallel, and constant along the parallel.

Eq 35. $$Length(\lambda_0,\lambda_n)=\int_{\lambda_0}^{\lambda_n}\rho(\varphi)\cos\varphi d\lambda=\rho(\varphi)\cos\varphi\left|\lambda_n-\lambda_0\right|$$

8.3 Length of a Curve

The definition of a curve on an ellipsoid (see ISO 19107 Geographic information — Spatial schema. A curve is defined by a set of segments $\text{points}=\{p_0=(\varphi_o,\lambda_0),\,p_1=(\varphi_1,\lambda_1)\ldots.,\,p_n=(\varphi_n,\lambda_n)\}$ between each pair subject to an interpolation mechanism. For a numeric integral, it is easier to approximations using linear segments. If the curve is a line string the original data points can be used, but other curves should be densified by the interpolated points will allow to use a linear interpolation. This linear approximation even for complex curves works towards the correct measures.

In general, for two points on curve work better if the Δφ if less than a quarter degree, and the difference between the line used in the numeric integration is relatively and is curve is quite small.

This suggests that a GIS metric system should include a "center point" function for each curve type, so that the line approximation can be used to support the accuracy needed for length digital integration.

For example, if we are dealing with $p_0=(\varphi_o,\lambda_0)$ and $p_1=(\varphi_1,\lambda_1)$ then there is a plane perpendicular to the line between p_0 and p_1 defines a surface perpendicular to the line between the two points, and the point on the curve that is in that surface e.g. $p_{0.5}$ between the two point that lies on the curve and is approximately half way between p_0 and p_1.Continuing along the curve and introducing new half distance points along the curve, the difference between data points become closer to each other on the curve. Once the approximate distance between any two sequential points on the curve are within a quarter degree in both latitude (φ) and longitude (λ). Which approximates the distance between the two points as:

Eq 36. $$\left|p_{i+1} - p_i\right| \cong \sqrt{\left(\frac{M(\varphi_i) + M(\varphi_{i+1})}{2}\right)^2 (\Delta\varphi_i)^2 + \left(\frac{\rho(\varphi_i) + \rho(\varphi_{i+1})}{2}\right)^2 (\Delta\lambda_i)^2}$$

In the equation below, the calculation of a curve length should be

$$\text{points} = \{p_0 = (\varphi_o, \lambda_0), p_1 = (\varphi_1, \lambda_1)...., p_n = (\varphi_n, \lambda_n)\}$$

$$\text{Curve } c(s) = (\varphi(s), \lambda(s))$$

Eq 37. $$\begin{aligned} \text{Lengtht}(p_0,p_n) &= \int_0^t \sqrt{\left(M(\varphi(s))\left(\frac{d\varphi}{ds}\right)\right)^2 + \left(\rho(\varphi(s))\left(\frac{d\lambda}{ds}\right)\right)^2}\, ds \\ &= \lim_{\substack{n\to\infty \\ \Delta\to 0}} \sum_{i=1}^{n} \sqrt{(M(\varphi_i))^2 (\Delta\varphi_i)^2 + (\rho(\varphi_i))^2 (\Delta\lambda_i)^2} \\ &= \lim_{\substack{n\to\infty \\ \Delta\to 0}} \sum_{i=1}^{n} \sqrt{\left(\frac{M(\varphi_i) + M(\varphi_{i-1})}{2}\right)^2 (\Delta\varphi_i)^2 + \left(\frac{\rho(\varphi_i) + \rho(\varphi_{i-1})}{2}\right)^2 (\Delta\lambda_i)^2} \end{aligned}$$

The two variations use the original Newton’s definition, and the later uses a trapezoid rule that makes a better local approximation because the ISO 19107 Geographic information — Spatial schema

Equations Eq 7 (curves) and Eq 8 (areas) are the integrals, and Eq 9 and Eq 10 are the numeric integrations of the integrals. The later approximations can be done to best that can be done in double precisions numbers. The tables below demonstrate this. See Table 2, Table 3 and Table 4.

Annex A Descriptive values of an oblate ellipsoid

A.1 Ellipsoidal constants.

The following equations describe the important information of ellipsoids including geodetic coordinates (φ,λ): $\frac{X^2}{a^2}+\frac{Y^2}{a^2}+\frac{Z^2}{b^2}=1$, where "a" is the semi-major axis (equatorial radius=6,378,137.0 m) and "b" is the semi-minor axis (the polar radius=6,356,752.314245). The inverse flattening is 298.257223563. Although all equations are valid for all ellipsoid, the numerical values are WGS84, the common spheroid for GPS.

Eq 38. Ellipsoid S^2 (surface) and D^3 (solid):

$$S^2 \Rightarrow \frac{X^2}{a^2}+\frac{Y^2}{a^2}+\frac{Z^2}{b^2}=1$$
$$D^3 \Rightarrow \frac{X^2}{a^2}+\frac{Y^2}{a^2}+\frac{Z^2}{b^2}\le 1$$

Eq 39. Radii:

$$a = 6,378,137.0 \text{ m} \quad \text{by definition}$$
$$b = 6,356,752.314245180 \text{ m} \quad \text{by calculation } b = a(1-f)$$

Eq 40. First eccentricity: $e=\sqrt{\frac{a^2-b^2}{a^2}}=2f-f^2$

Eq 41. Second eccentricity: $e'=\sqrt{\frac{a^2-b^2}{b^2}}$

Eq 42. Flattening: $f^{-1}=\frac{1}{f}=298.257223563=\left(\frac{a-b}{a}\right)^{-1}=\frac{a}{a-b}$

Eq 43. First flattening: $f=\frac{a-b}{a}$ =298.257223563

Eq 44. Inverse Flattening: $f^{-1}=\frac{1}{f}$

Eq 45. Second flattening: $f'=\frac{a-b}{b}$

A.2 Geodetic Ellipsoidal coordinates.

This clause shows that the standard mapping between geodetic (φ, λ) and ECEF (X,Y,Z) are consistent.

Eq 46. The ellipsoid surface: $\frac{X^2}{a^2}+\frac{Y^2}{a^2}+\frac{Z^2}{b^2}=1$

Eq 47. Length of Normal: $N(\varphi)=\frac{a}{\sqrt{1-e^2\sin^2\varphi}}=\frac{a^2}{\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}}$)

Eq 48. Radius of Meridian: $M(\varphi)=\frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}}$

Eq 49. X in geodetic: $X=N(\varphi)\cos\varphi\cos\lambda=\frac{a\cos\varphi\cos\lambda}{\sqrt{1-e^2\sin^2\varphi}}=\frac{a^2\cos\varphi\cos\lambda}{\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}}$

Eq 50. Y in geodetic: $Y=N(\varphi)\cos\varphi\sin\lambda=\frac{a\cos\varphi\sin\lambda}{\sqrt{1-e^2\sin^2\varphi}}=\frac{a^2\cos\varphi\sin\lambda}{\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}}$

Eq 51. Z in geodetic: $Z=N(\varphi)(1\text{-}e^2)\sin\varphi=\frac{a(1\text{-}e^2)\sin\varphi}{\sqrt{1-e^2\sin^2\varphi}}=\frac{b^2\sin\varphi}{\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}}$

Eq 52. Radius of parallel:

$$\begin{aligned}\rho(\varphi)&=\sqrt{X^2+Y^2}\\&=\sqrt{N^2(\varphi)\cos^2\varphi(\sin^2\varphi+\cos^2\varphi)}\\&=N(\varphi)\cos\varphi=\frac{a\cos\varphi}{\sqrt{1-e^2\sin^2\varphi}}\end{aligned}$$

Eq 53. Coordinates:

$$\begin{aligned}X&=\rho(\varphi)\cos\lambda=\frac{a^2\cos\varphi\cos\lambda}{\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}}=\frac{a\cos\varphi\cos\lambda}{\sqrt{1-e^2\sin^2\varphi}}\\Y&=\rho(\varphi)\sin\lambda=\frac{a^2\cos\varphi\sin\lambda}{\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}}=\frac{a\cos\varphi\sin\lambda}{\sqrt{1-e^2\sin^2\varphi}}\\Z&=\frac{b^2\sin\varphi}{\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}}=\frac{a(1\text{-}e^2)\sin\varphi}{\sqrt{1-e^2\sin^2\varphi}}\end{aligned}$$

Thus, satisfying the ellipsoidal equation:

Eq 54.

$$
\begin{aligned}
\frac{X^2+Y^2}{a^2}+\frac{Z^2}{b^2} &= \frac{1}{a^2}\left(\frac{a^4\cos^2\varphi\left(\cos^2\lambda+\sin^2\lambda\right)}{a^2\cos^2\varphi+b^2\sin^2\varphi}\right)+\frac{1}{b^2}\left(\frac{b^4\sin^2\varphi}{a^2\cos^2\varphi+b^2\sin^2\varphi}\right)\\
&= \left(\frac{a^2\cos^2\varphi}{a^2\cos^2\varphi+b^2\sin^2\varphi}\right)+\left(\frac{b^2\sin^2\varphi}{a^2\cos^2\varphi+b^2\sin^2\varphi}\right)\\
&= \frac{a^2\cos^2\varphi+b^2\sin^2\varphi}{a^2\cos^2\varphi+b^2\sin^2\varphi}\\
&= 1
\end{aligned}
$$

Burkholder [4], Bomford [3], Clynch [6], Hotine [16], IOGP[19], Jekeli [20], Krakiwsky and Thomson [21], Ligas [22], Panigrahi [26], Torge [31].

A.3 Geodetic Metric, Latitude (φ) And Longitude (λ)

Using classical mathematical trigonometry, the above equations can be expressed, using geodetic latitude φ and longitude λ. Taking the equations X, Y, Z expressed functions in geocentric coordinates, (φ,λ). This is essentially the creation of the matrix of transformations J $\left(\frac{X,Y,Z}{\varphi,\lambda}\right)$. Below, using geodetic latitude using the equations in the ECEF (GSDM), we calculate the X,Y, Z coordinates as functions of latitude and longitude, (φ,λ). Taking derivatives with respect to both φ and λ, we get the radius of curvature for the axes for both φ and λ. The values come out squared, because what is derived is the dot products of the tangent vector, e.g. the squares of the radii of curvature.

Eq 55. Ellipsoid surface:

$$\frac{X^2}{a^2}+\frac{Y^2}{a^2}+\frac{Z^2}{b^2}=1$$

Eq 56. Length of normal:

$$N(\varphi)=\frac{a}{\sqrt{1-e^2\sin^2\varphi}}=\frac{a^2}{\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}}$$

$$N'(\varphi)=\frac{ae^2\sin\varphi\cos\varphi}{\left(1-e^2\sin^2\varphi\right)^{3/2}}=\frac{a^4e^2\sin\varphi\cos\varphi}{\left(a^2\cos^2\varphi+b^2\sin^2\varphi\right)^{3/2}}$$

Eq 57. Radius of Parallel:

$$\rho(\varphi)=N(\varphi)\cos\varphi=\frac{a\cos\varphi}{\sqrt{1-e^2\sin^2\varphi}}$$

Eq 58. Radius of Meridian:

$$M(\varphi)=\frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}}$$

Eq 59. XYZ to elliptical:
$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \dfrac{a^2\cos\varphi\cos\lambda}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}} = N(\varphi)\cos\varphi\cos\lambda = \dfrac{a\cos\varphi\cos\lambda}{\sqrt{1-e^2\sin^2\varphi}} \\ \dfrac{a^2\cos\varphi\sin\lambda}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}} = N(\varphi)\cos\varphi\sin\lambda = \dfrac{a\cos\varphi\sin\lambda}{\sqrt{1-e^2\sin^2\varphi}} \\ \dfrac{b^2\sin\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}} = N(\varphi)\left(b^2/a^2\right)\sin\varphi = \dfrac{a\left(1-e^2\right)\sin\varphi}{\sqrt{1-e^2\sin^2\varphi}} \end{bmatrix}$$

This creates the components for the first fundamental form, which is a vector inner product for $\vec{u}$ and $\vec{v}$ in the vector space for the coordinates$(\boldsymbol{\varphi}, \boldsymbol{\lambda})$ whose vectors spanned by $(\vec{\boldsymbol{v}}, \vec{\boldsymbol{u}})$as defined above. In other words, these are vectors for the coordinate space of $(\boldsymbol{\varphi}, \boldsymbol{\lambda})$ i.e. the reference ellipsoid. The inner product also implies that any vectors in the φ-direction at a point is always perpendicular to vectors in the λ-direction.

Geocentric measures are measured along a line from the center of the ellipsoid (origin of the coordinate system) to the point. Geodetic latitude makes things a bit more complex. A geocentric latitude is slope of the upward normal from the surface of the ellipsoid which is equal to the geocentric latitude at the equator and the poles.

Similar calculations for geodetic latitude are a bit more complicated. The equations for this form has been, often with slightly different variable names, around for quite a long time, See Bomford [3] (at least in the 3rd and later editions), Burkholder [4], Hotine [16], IOGP[19], Jekeli [20], Krakiwsky and Thomson [21], Ligas [22], Panigrahi [26], Torge [31], Clynch [6], [7] and [8]. The differences in the various versions of the model were essentially in the choice of variable names, and the details on the calculations.

$$\mathrm{J}\left(\frac{X,Y,Z}{\varphi,\lambda}\right) = \left[\frac{\partial\vec{X}}{\partial\varphi} = d\varphi \quad \frac{\partial\vec{X}}{\partial\lambda} = d\lambda\right]$$

Eq 60. Reimann metric:
$$= \left[\frac{\partial}{\partial\varphi}\begin{bmatrix} \mathrm{N}(\varphi)\cos\varphi\cos\lambda \\ \mathrm{N}(\varphi)\cos\varphi\sin\lambda \\ \mathrm{N}(\varphi)(1-e^2)\sin\varphi \end{bmatrix} \quad \frac{\partial}{\partial\lambda}\begin{bmatrix} \mathrm{N}(\varphi)\cos\varphi\cos\lambda \\ \mathrm{N}(\varphi)\cos\varphi\sin\lambda \\ \mathrm{N}(\varphi)(1-e^2)\sin\varphi \end{bmatrix}\right]$$

$$= \begin{bmatrix} \left(\mathrm{N}'(\varphi)\cos\varphi - \mathrm{N}(\varphi)\sin\varphi\right)\cos\lambda & -\mathrm{N}(\varphi)\cos\varphi\sin\lambda \\ \left(\mathrm{N}'(\varphi)\cos\varphi - \mathrm{N}(\varphi)\sin\varphi\right)\sin\lambda & \mathrm{N}(\varphi)\cos\varphi\cos\lambda \\ \left(1-e^2\right)\left(\mathrm{N}'(\varphi)\sin\varphi + \mathrm{N}(\varphi)\cos\varphi\right) & 0 \end{bmatrix}$$

Eq 61. Radii φ,λ:

$$\vec{u}_1 = \frac{\partial \vec{X}}{\partial \varphi} = \begin{bmatrix} \left(\mathrm{N}'(\varphi)\cos\varphi - \mathrm{N}(\varphi)\sin\varphi\right)\cos\lambda \\ \left(\mathrm{N}'(\varphi)\cos\varphi - \mathrm{N}(\varphi)\sin\varphi\right)\sin\lambda \\ (1-e^2)\left(\mathrm{N}'(\varphi)\sin\varphi + \mathrm{N}(\varphi)\cos\varphi\right) \end{bmatrix};$$

$$\vec{u}_2 = \frac{\partial \vec{X}}{\partial \lambda} = \begin{bmatrix} -\mathrm{N}(\varphi)\cos\varphi\sin\lambda \\ \mathrm{N}(\varphi)\cos\varphi\cos\lambda \\ 0 \end{bmatrix}$$

First fundamental form for geodetic coordinates (φ, λ):

Eq 62. *E(φ)*:

$$E(\varphi) = a_{1,1} = \left(\mathrm{N}'(\varphi)\cos\varphi - \mathrm{N}(\varphi)\sin\varphi\right)^2 + \left(\frac{b^2}{a^2}\right)^2\left(\mathrm{N}'(\varphi)\sin\varphi + \mathrm{N}(\varphi)\cos\varphi\right)^2$$

$$\sqrt{E(\varphi)} = M(\varphi) = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}} = a(1-e^2)\left(1-e^2\sin^2\varphi\right)^{-3/2}$$

The equality $E(\varphi) = M^2(\varphi)$ has been checked numerically for $\pm 90^\circ$. An algebraic proof has been difficult to find. In essence, the calculations for $\sqrt{E(\varphi)}$ and $M(\varphi)$ are both valid calculations for a meridian's radius of curvature, e.g. $u_1 \bullet u_1 = |u_1|^2 = (\text{radius of curvature})^2$.

Eq 63. $F(\varphi) = 0 = a_{1,2} = a_{2,1} = 0;\ \left[a_{i,j}\right] = \left[\vec{u}_i \bullet \vec{u}_j\right];\ \vec{u}_i \bullet \vec{u}_j = \vec{u}_i\left[a_{i,j}\right]\vec{u}_j^{\,t}$

Eq 64. $G(\varphi) = a_{2,2} = \left(\mathrm{N}(\varphi)\right)^2\cos^2\varphi\left(\sin^2\lambda + \cos^2\lambda\right) = \mathrm{N}^2(\varphi)\cos^2\varphi = \left(\mathrm{N}(\varphi)\cos\varphi\right)^2 = \rho^2(\varphi)$

The Riemannian metric is:

Eq 65.

$$\left[a_{i,j}\right] = \begin{bmatrix} \left(\mathrm{N}'(\varphi)\cos\varphi + \mathrm{N}(\varphi)\sin\varphi\right)^2 + \frac{b^4}{a^4}\left(\mathrm{N}'(\varphi)\sin\varphi + \mathrm{N}(\varphi)\cos\varphi\right)^2 & 0 \\ 0 & \mathrm{N}^2(\varphi)\cos^2\varphi \end{bmatrix}$$

$$= \begin{bmatrix} M^2(\varphi) & 0 \\ 0 & \mathrm{N}^2(\varphi)\cos^2\varphi \end{bmatrix}$$

A.4 Geocentric Metric - Latitude (ψ), Longitude (λ)

The equation of the surface that uses latitude and longitude as both central angles of the ellipsoid would be as follows:

Eq 66. Ellipsoid: $$\frac{X^2+Y^2}{a^2}+\frac{Z^2}{b^2}=1$$

Using classical mathematical trigonometry, the above equations can be expressed, using geocentric latitude φ_c and longitude λ. Taking the equations X, Y, Z expressed functions in geocentric or geodetic coordinates, (φ,λ). This is essentially the create the radius of curvature along the φ (meridians) and λ (parallels) coordinate lines. The metric uses the squares of the radii of curvature.

Geocentric ellipsoidal coordinates, $S^2(\psi,\lambda) \subset \mathbb{E}^{\mathbf{3}}(X,Y,Z)$ where:

Eq 67. Geocentric: $$\vec{X}=\begin{bmatrix} a\cos\psi\cos\lambda \\ a\cos\psi\sin\lambda \\ b\sin\psi \end{bmatrix}$$

Eq 68. Geocentric coordinates:

$$\mathrm{J}\left(\frac{X,Y,Z}{\varphi,\lambda}\right)=\begin{bmatrix} \frac{\partial\vec{X}}{\partial\psi}=d\psi & \frac{\partial\vec{X}}{\partial\lambda}=d\lambda \end{bmatrix}$$
$$=\begin{bmatrix} \frac{\partial}{\partial\psi}\begin{bmatrix} a\cos\psi\sin\lambda \\ a\cos\psi\cos\lambda \\ b\sin\psi \end{bmatrix} & \frac{\partial}{\partial\lambda}\begin{bmatrix} a\cos\psi\sin\lambda \\ a\cos\psi\cos\lambda \\ b\sin\psi \end{bmatrix} \end{bmatrix}$$
$$=\begin{bmatrix} -a\sin\psi\sin\lambda & a\cos\psi\cos\lambda \\ -a\sin\psi\cos\lambda & -a\cos\psi\sin\lambda \\ b\cos\psi & 0 \end{bmatrix}$$

Eq 69. $$\vec{u}_1=\frac{\partial\vec{X}}{\partial\psi}=[-a\sin\psi\cos\lambda,-a\sin\psi\sin\lambda,b\cos\psi]$$

Eq 70. $$\vec{u}_2=\frac{\partial\vec{X}}{\partial\lambda}=[-a\cos\psi\sin\lambda,a\cos\psi\cos\lambda,0]$$

Eq 71.
$$\vec{u}_1\bullet\vec{u}_1=E=a_{11}=a^2\sin^2\psi+b^2\cos^2\psi$$
$$\vec{u}_1\bullet\vec{u}_2=F=a_{12}=a_{21}=0$$
$$\vec{u}_2\bullet\vec{u}_2=G=a_{22}=a^2\cos^2\psi$$

A.5 First Fundamental Form Using Geocentric Latitude "ψ":

Eq 72. Geocentric Metric $[a_{ij}]=\begin{bmatrix} E & F \\ F & G \end{bmatrix}=\begin{bmatrix} a^2\sin^2\psi+b^2\cos^2\psi & 0 \\ 0 & a^2\cos^2\psi \end{bmatrix}$

This creates the pieces for the first fundamental form, which is a vector inner product for $\vec{u}$ and $\vec{v}$ in the vector space for the coordinates(ψ, λ) whose vectors spanned by $(\vec{v}, \vec{u})$ as defined above. In other words, these are vectors for the coordinate space of (ψ, λ) i.e. the reference ellipsoid. The inner product $[g_{ij}]$ also implies that any vectors in the ψ-direction at a point is always perpendicular to vectors in the λ-direction.

A.6 Geocentric Metric on a sphere - Latitude (ψ), Longitude (λ)

If $a = b$ then the ellipsoid is a sphere $(a = r = b)$ and the first fundamental of the sphere using geocentric latitude "ψ" and longitude "λ" is:

Eq 73. Geocentric Metric on a spherer: $\begin{bmatrix} E & F \\ F & G \end{bmatrix}=\left[a_{ij}\right]=\begin{bmatrix} r^2 & 0 \\ 0 & r^2\cos\psi \end{bmatrix}$

On the sphere, the geocentric "ψ" and geodetic "φ" latitude are the same because the circle's eccentricity in 0. It should be remembered that for a sphere, r=a=b, and any metric will work. For example, the next equation can be used to calculate this by doing just that:

Eq 74. $sin^2\,\psi + cos^2\,\psi = 1$

Geocentric measures are measured along a line from the center of the ellipsoid (origin of the coordinate system) to the point. Geodetic latitude makes things a bit more complex. A geocentric latitude is the slope of the upward normal from the surface of the ellipsoid which is equal to the geocentric latitude at the equator and the poles.

In all cases, the integral for length of a curve in equation (70) and the area or a region W in equations in (74) are valid for each fundamental form for the variables (in these cases "latitude", "ψ" or "φ", and longitude "λ") used in the calculation of the form.

A.7 Three-Dimensional Geodetic Metric (φ, λ, h)

This clause shows that the standard mapping between geodetic (φ, λ) and ECEF (X, Y, Z) are consistent.

Eq 75. X in geodetic: $X = \left(N\left(\varphi\right)+h\right)\cos\varphi\cos\lambda$

Eq 76. Y in geodetic: $Y = \left(N\left(\varphi\right)+h\right)\cos\varphi\sin\lambda$

Eq 77. Z in geodetic: $$Z = \left(N(\varphi)(1\text{-}e^2) + h\right)\sin\varphi$$

Eq 78. Radius of parallel: $$\rho = \sqrt{X^2 + Y^2} = \sqrt{\left(N(\varphi)+h\right)^2 \cos^2\varphi} = \left(N(\varphi)+h\right)\cos\varphi$$

Eq 79. Coordinates:
$$\begin{aligned} X &= \rho\cos\lambda = \left(N(\varphi)+h\right)\cos\varphi\cos\lambda, \\ Y &= \rho\sin\lambda = \left(N(\varphi)+h\right)\cos\varphi\sin\lambda \\ Z &= \left(N(\varphi)(1\text{-}e^2)+h\right)\sin\varphi \end{aligned}$$

Using classical mathematical trigonometry, the above equations can be expressed, using geodetic latitude φ and longitude λ. Taking the equations X, Y, Z expressed functions in geocentric coordinates, (φ, λ). This is essentially the creation of the Jacobian of the coordinate transformation between ECEF (X, Y, Z) and latitude - longitude (φ, λ).

Eq 80.
$$N(\varphi) = \frac{a}{\sqrt{1-e^2\sin^2\varphi}} \qquad N'(\varphi) = \frac{ae^2\sin\varphi\cos\varphi}{\left(1-e^2\sin^2\varphi\right)^{3/2}}$$

Eq 81.
$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \left(N(\varphi)+h\right)\cos\varphi\cos\lambda \\ \left(N(\varphi)+h\right)\cos\varphi\sin\lambda \\ \left(N(\varphi)+h\right)(1-e^2)\sin\varphi \end{bmatrix}$$

This creates the components for the first fundamental form, which is a vector inner product for $\vec{u}$ and $\vec{v}$ in the vector space for the coordinates (φ,λ) whose vectors spanned by $(\vec{\boldsymbol{v}}, \vec{\boldsymbol{u}})$as defined above. In other words, these are vectors for the coordinate space of $(\boldsymbol{\varphi}, \boldsymbol{\lambda})$ i.e. the reference ellipsoid. The inner product also implies that any vectors in the φ-direction at a point is always perpendicular to vectors in the λ-direction.

Geocentric measures are measured along a line from the center of the ellipsoid (origin of the coordinate system) to the point. Geodetic latitude makes things a bit more complex. A geocentric latitude is slope of the upward normal from the surface of the ellipsoid which is equal to the geocentric latitude at the equator and the poles.

Similar calculations for geodetic latitude are a bit more complicated. The equations for this form has been, often with slightly different variable names, around for quite a long time, See Bomford [3] (at least in the 3rd and later editions), Burkholder [4], Hotine [16], IOGP[19], Jekeli [20], , Ligas [22], Panigrahi [26], Torge [31], Clynch [6], [7] and [8]. The differences in the various versions of the model were essentially in the choice of variable names, and the details on the calculations.

The first step in each case is to determine the local perpendicular and the length of the this principal normal as a function of latitude N(φ), with respect to the east-west "longitude lines".

Eq 82.
$$J\left(\frac{X,Y,Z}{\varphi,\lambda,h}\right)=\begin{bmatrix}\frac{\partial\vec{X}}{\partial\varphi}=d\varphi & \frac{\partial\vec{X}}{\partial\lambda}=d\lambda & \frac{\partial\vec{X}}{\partial h}=dh\end{bmatrix}$$

Eq 83.
$$J\left(\frac{X,Y,Z}{\varphi,\lambda,h}\right)=\begin{bmatrix}\vec{u}_1=d\varphi=\frac{\partial}{\partial\varphi}\begin{bmatrix}\left(N(\varphi)+h\right)\cos\varphi\cos\lambda\\ \left(N(\varphi)+h\right)\cos\varphi\sin\lambda\\ \left(N(\varphi)+h\right)(1-e^2)\sin\varphi\end{bmatrix}\\ \vec{u}_2=d\lambda=\frac{\partial}{\partial\lambda}\begin{bmatrix}\left(N(\varphi)+h\right)\cos\varphi\cos\lambda\\ \left(N(\varphi)+h\right)\cos\varphi\sin\lambda\\ \left(N(\varphi)+h\right)(1-e^2)\sin\varphi\end{bmatrix}\\ \vec{u}_3=dh=\frac{\partial}{\partial h}\begin{bmatrix}\left(N(\varphi)+h\right)\cos\varphi\cos\lambda\\ \left(N(\varphi)+h\right)\cos\varphi\sin\lambda\\ \left(N(\varphi)+h\right)(1-e^2)\sin\varphi\end{bmatrix}\end{bmatrix}$$

Eq 84.
$$\begin{bmatrix}d\varphi & d\lambda & dh\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}N'(\varphi)\cos\varphi\cos\lambda-\left(N(\varphi)+h\right)\sin\varphi\cos\lambda\\ N'(\varphi)\cos\varphi\sin\lambda-\left(N(\varphi)+h\right)\sin\varphi\sin\lambda\\ N'(\varphi)(1-e^2)\sin\varphi+\left(N(\varphi)+h\right)(1-e^2)\cos\varphi\end{bmatrix}\\ \begin{bmatrix}-\left(N(\varphi)+h\right)\cos\varphi\sin\lambda\\ \left(N(\varphi)+h\right)\cos\varphi\cos\lambda\\ 0\end{bmatrix}\\ \begin{bmatrix}\cos\varphi\cos\lambda\\ \cos\varphi\sin\lambda\\ (1-e^2)\sin\varphi\end{bmatrix}\end{bmatrix}$$

Eq 85.
$$\vec{u}_1=\frac{\partial\vec{X}}{\partial\varphi}=\begin{bmatrix}\left(\mathrm{N}'(\varphi)\cos\varphi-\left(N(\varphi)+h\right)\sin\varphi\right)\cos\lambda\\ \left(\mathrm{N}'(\varphi)\cos\varphi-\left(N(\varphi)+h\right)\sin\varphi\right)\sin\lambda\\ (1-e^2)\left(\mathrm{N}'(\varphi)\sin\varphi+\left(N(\varphi)+h\right)\cos\varphi\right)\end{bmatrix}$$

Eq 86. $$\vec{u}_2 = \frac{\partial \vec{X}}{\partial \lambda} = \begin{bmatrix} -\left(N(\varphi)+h\right)\cos\varphi\sin\lambda \\ \left(N(\varphi)+h\right)\cos\varphi\cos\lambda \\ 0 \end{bmatrix}$$

Eq 87. $$\vec{u}_3 = \frac{\partial \vec{X}}{\partial h} = \begin{bmatrix} \cos\varphi\cos\lambda \\ \cos\varphi\sin\lambda \\ \left(1-e^2\right)\sin\varphi \end{bmatrix}$$

First fundamental form for geodetic coordinates (φ,λ):

Eq 88. $$\begin{aligned} a_{1,1} &= \left(\mathrm{N}'(\varphi)\cos\varphi - \left(\mathrm{N}(\varphi)+h\right)\sin\varphi\right)^2 + \left(1-e^2\right)^2\left(\mathrm{N}'(\varphi)\sin\varphi + \left(\mathrm{N}(\varphi)+h\right)\cos\varphi\right)^2 = m^2(\varphi) \\ a_{2,2} &= \left(\mathrm{N}(\varphi)\right)^2\cos^2\varphi\left(\sin^2\lambda+\cos^2\lambda\right) = \mathrm{N}^2(\varphi)\cos^2\varphi = \left(\mathrm{N}(\varphi)\cos\varphi\right)^2 = \rho^2(\varphi) \\ a_{3,3} &= \cos^2\varphi + \frac{b^4}{a^4}\sin^2\varphi = \cos^2\varphi + (1-e^2)^2\sin^2\varphi \\ a_{i,j} &= \vec{u}_i \bullet \vec{u}_j = 0 \text{ iff } i \neq j \end{aligned}$$

Eq 89. $$\left[a_{i,j}\right] = \begin{bmatrix} \left(\mathrm{N}'(\varphi)\cos\varphi + \mathrm{N}(\varphi)\sin\varphi\right)^2 + \frac{b^4}{a^4}\left(\mathrm{N}'(\varphi)\sin\varphi + \mathrm{N}(\varphi)\cos\varphi\right)^2 & 0 & 0 \\ 0 & \mathrm{N}^2(\varphi)\cos^2\varphi & 0 \\ 0 & 0 & \cos^2\varphi + \frac{b^4}{a^4}\sin^2\varphi \end{bmatrix}$$

Annex B Metric Integrals and Numeric Approximations

B.1 Length and Area Integrals.

This section defines the Riemannian metrics, which are integrals derived from Gauss's work and Newton's calculus. In these cases, the calculus is rather difficult and usually does not lead to simple integrations in closed form. This means there are not simple formulas as in the Pythagorean metric. This means that the best viable solutions are numeric integrations which are approximations based on the simple summations that show-up in the basic definitions of an integral an area under a curve.

This is quite easy to understand, an integral $\int_a^b f(t)dt$ is the area under the curve $f(t)$ between the axis and the curve between $t = a$ and $t = b$. These solutions in fairly simple loops that reiterate summations that approximate this area. The key to getting a good approximation is to use longer and longer summations based on smaller and smaller intervals for latitude and longitude e.g. shorter Δφ's and Δλ's (see below).

For geodetic coordinates, the following metric functions apply:

Eq 90. Meridian radius:

$$E^2(\varphi) = \left(\mathrm{N}'(\varphi)\cos\varphi + \mathrm{N}(\varphi)\sin\varphi\right)^2 + \frac{b^4}{a^4}\left(\mathrm{N}'(\varphi)\sin\varphi + \mathrm{N}(\varphi)\cos\varphi\right)^2$$

$$= M^2(\varphi)$$

$$M(\rho) = \frac{a(1-e^2)}{\left(1-e^2\sin^2\varphi\right)^{3/2}} = \sqrt{E(\varphi)}$$

Eq 91. Parallel radius:

$$\mathrm{G}(\varphi) = \mathrm{N}(\varphi)^2\cos^2\varphi = \rho^2(\varphi) = (N(\varphi)\cos\varphi)^2$$

$$\rho(\varphi) = N(\varphi)\cos\varphi = \sqrt{G(\varphi)}$$

The relationship between $E(\varphi) = m^2(\varphi)$ is addressed in Annex B.2. A formal proof is yet to be found, but extensive numeric testing indicates $E(\varphi)=\mathrm{m}^2(\varphi)$ are equal for all values of "φ". The most common set of parameters are from WGS84:

B.2 The Curve Length Integral and the Numeric Alternative

Integration in calculus is inherently difficult, because unlike derivatives, there is often not a fixed procedure. In the use of the integrals in the above narrative are a line integral to calculate the length of a geometry (curve) (E, F and G are distances in meters, and φ and λ are angles represented in radians, i.e. no unit (radians are a ratio and inherently unitless, e.g. π=180˚,1˚=π/180=0.0174532925199). In all cases in this paper φ and λ lines are orthogonal everywhere, so F≡0.

To understand what follows, we need to understand the relationship between φ, λ, and t on a curve c: $c(t) = (\varphi(t), \lambda(t))$; we have 3 sequences for $(\varphi_i, \lambda_i, t_i)$ where $c(t_i) = (\varphi(t_i), \lambda(t_i)) = (\varphi_i, \lambda_i)$. Assuming the Δφ, and Δλ, are small enough to keep the radii of curvatures within a small area so that they are "fairly smooth".

Eq 92. *Curve* $= \begin{bmatrix} \mathrm{c(t)} = \{\mathrm{c}(\mathrm{t}_i) = (\varphi_i, \lambda_i) \mid i = 0,1,2,3\ldots n;\} \\ \Delta\varphi_i = |\varphi_i - \varphi_{i-1}|; \Delta\lambda_i = |\lambda_i - \lambda_{i-1}| \end{bmatrix}$

$$L_c = \int_a^b \sqrt{E(\varphi)\left(\frac{d\varphi}{dt}\right)^2 + G(\varphi)\left(\frac{d\lambda}{dt}\right)^2}\, dt$$

Eq 93. *Length Integral*: $L_c = \lim_{\substack{n\to\infty \\ \Delta\to 0}} \sum_{i=1}^{n} \sqrt{E\left(\frac{\varphi_i + \varphi_{i-1}}{2}\right)(\Delta\varphi_i)^2 + G\left(\frac{\varphi_i + \varphi_{i-1}}{2}\right)(\Delta\lambda_i)^2}$

$$= \lim_{\substack{n\to\infty \\ \Delta\to 0}} \sum_{i=1}^{n} \sqrt{M^2\left(\frac{\varphi_i + \varphi_{i-1}}{2}\right)(\Delta\varphi_i)^2 + \rho^2\left(\frac{\varphi_i + \varphi_{i-1}}{2}\right)(\Delta\lambda_i)^2}$$

As seen above, as the Δφ and Δλ grow smaller and more numerous, the numeric calculations use the Pythagorean formula which works best if the square bounded by $\Delta\varphi_i$, and $\Delta\lambda_i$ is on the order of a fraction of degree for both for both latitude and longitude. This works because in small areas the classical geometry works in what we refer to as engineering diagrams or plans. In this approximation we used the midpoint value for *E* and *G* between φ_i and φ_{i-1}. In the examples in

Each of the possible options do something equivalent, approximating the area of the polygon or trapezoid under the function in the integral by multiplying the width of the represented by the values of Δφ and Δλ, where the function functions *E* and G in that supply the "meters per radian" for the locality of the arc.

Annex B also contains the mechanism for these integrals to be calculated using summations. This derives directly from the definition of an integral as the limit of longer and longer sums.

The idea is to divide the subdivisions into smaller and smaller ones, for example, by placing a new value for "*s*" between each existing pair, doubling the number of intervals and halving each interval by inserting a midpoint.

Eq 94. *Curve in* (*φ,λ*)

$$\begin{aligned} a &= t_0, t_1, \ldots, t_n = b \\ \Delta t_i &= t_i - t_{i-1} \\ (\varphi_i, \lambda_i) &= (\varphi(t_i), \lambda(t_i)) \\ \Delta\varphi_i &= \varphi_i - \varphi_{i-1} \\ \Delta\lambda_i &= \lambda_i - \lambda_{i-1} \end{aligned}$$

$$t_i = t'_{2i} \text{ and } t'_{2i+1} = \frac{t_i + t_{i+1}}{2} = \frac{t'_{2i} + t'_{2i+2}}{2}$$

$$\Delta L_i = \left(\left(\frac{M(\varphi_i) + M(\varphi_{i-1})}{2} \right)^2 \Delta\varphi_i^2 + \left(\frac{\rho(\lambda_i) + \rho(\lambda_{i-1})}{2} \right)^2 \Delta\lambda_i^2 \right)^{\frac{1}{2}}$$

Eq 95. Length of curve:

$$L_c = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \Delta L_i \right)$$

$$\Delta A_c = \left(\left(\frac{M(\varphi_i) + M(\varphi_{i-1})}{2} \right) \Delta\varphi_i \times \left(\frac{\rho(\lambda_i) + \rho(\lambda_{i-1})}{2} \right) \Delta\lambda_i \right)$$

$$A_c = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \Delta A_{ci} \right)$$

Each this iterative summation will get closer to the actual L_c if sufficiently accurate number formats are used (64-bit double-precision at the least). The usual approach might be using the number of nodes used in the digital curve (as defined in ISO 19107 Geographic information — Spatial schema, and half each interval length (doubling the count) each time until subsequent difference in subsequent iterations are within the desired error budget.

B.3 The Surface Area Integral

The surface area integral calculates the area of a subsurface of the reference surface (ellipsoid), which is simple product of the length times the width in $\sqrt{\boldsymbol{E(\varphi)G(\varphi)}}\boldsymbol{\Delta\varphi\Delta\lambda}$. Note that in both cases ($\sqrt{\boldsymbol{E(\varphi)}}\Delta\varphi$ and $\sqrt{\boldsymbol{G(\varphi)}}\Delta\lambda$) gives you the number of meters in a width in Δφ direction and a length in Δλ direction respectively. The two delta-angles are in radians, and the *E* and *G* parts just change angles to meters.

So, in a way the Riemannian measures of length and area depend on the usual Euclidean measures, good as long as the values of Δφ or Δλ are sufficiently small for the purpose. In a truly echo of history, what we are doing is exactly that which the 3rd century BC Greeks were using to approximate π by creating polygons to closely approximate the length or surface of the spheroid or circle. The area integral for the geodetic fundamental forms in this paper the area integral is (recall that F=0).

Eq 96. Area integral

$$A_S = \iint_W \sqrt{E(\varphi)G(\varphi)}\, d\varphi d\lambda = \iint_W \sqrt{E(\varphi)}\sqrt{G(\varphi)} d\varphi d\lambda$$

$$\sqrt{E(\varphi)} = M(\varphi) = a(1-e^2)(1-e^2\sin^2\varphi)^{-3/2}$$

$$\sqrt{G(\varphi)} = N(\varphi)|\cos\varphi|$$

The summation approximations for this integral break the area into square using subsets of the φ and λ squares. So, if the latitude is between $\varphi_0,\ldots\varphi_n$, and $\lambda_0,\ldots\lambda_n$:

The difference between the line integral and the area integral is the dimension of the summation sections. There are alternatives to what is below, but the same idea may be used with various methods of getting to smaller and smaller polygons.

1. Start with the minimum bounding rectangle. It is a square box that contains the entire area.
2. Slice all polygons into smaller polygons; any way that works, rectangles and triangles seem to work best. Throw out any that do not overlap the area $\boldsymbol{A}$.
3. For each remaining polygon measure the area in degrees in latitude and longitude. This is actually a unitless area, the units come from the integrand (the function between the long S "∫" and the variable differentials "dφdλ", which is in square meters (which is the way the derivations were made above, $\sqrt{E(\varphi)}$, and $\sqrt{G(\varphi)}$ are meters, so EG is therefore square meters).
4. Take the centroid each of the polygons (easy on squares and triangles); call it "p", and calculate an integrand at "p" inside the polygon preferably near the center, e.g. $\sqrt{E(p)G(p)}$ and multiply it by the "radian area" of the "dφdλ" polygon, e.g. $\sqrt{E(p)G(p)}\Delta\varphi\Delta\lambda = m(\varphi)\rho(\varphi)\Delta\varphi\Delta\lambda$ if it is the original still the original angular rectangle.
5. Sum all of these and keep it as the approximate area (recall, it is a number of square meters).
6. If the last 2 answers were very close within each other of the intended accuracy limit, stop and report the area at the integral value
7. If you did not stop at the last step, return step 2 and repeat.

B.4 The Integrals and their Numeric Approximations

Eq 97. The length integral:
$$L_c = \int_a^b \sqrt{E(\varphi)\left(\frac{d\varphi}{dt}\right)^2 + G(\varphi)\left(\frac{d\lambda}{dt}\right)^2}\,dt$$

The idea is that the curve in question is written as a function of "t" (think of time moving onward). So, our curve "c" is a function of "t" which keeps track or latitude, φ and longitude λ, so we write:

Eq 98. Derivtive of curve:
$$\left[c(t_i) = (\varphi(t_i), \lambda(t_i))\right] \Rightarrow \left[\frac{dc}{dt} = \frac{d}{dt}c(t_i) = (\frac{d}{dt}\varphi(t_i), \frac{d}{dt}\lambda(t_i))\right]$$

Which basically says that the derivative of a curve is the tangent vector (in physics, it is the velocity vector). Therefore, using the equation, the function (for velocity) it the square root of the functions from the first fundament form and an approximation of the vectors. In the geometry specification (ISO 19107) defines curves as a sequence of points: $\{\vec{\boldsymbol{p}}_{\boldsymbol{i}} = (\boldsymbol{\varphi}_{\boldsymbol{i}}, \boldsymbol{\lambda}_{\boldsymbol{i}}) | \boldsymbol{i} = \boldsymbol{0}., \boldsymbol{n}\}$ and {interpolation='curve type'} such that the curve between $\vec{\boldsymbol{p}}_{\boldsymbol{i-1}}$ and $\vec{\boldsymbol{p}}_{\boldsymbol{i}}$:

Eq 99. Generic curve
$$c(t_i) = (\varphi_i, \lambda_i),\ \text{i=0,1,2,...,n}$$
$$\Delta\varphi_i = \varphi_i - \varphi_{i-1}\ ;$$
$$\Delta\lambda_i = \lambda_i - \lambda_{i-1}\ \text{and}\ \frac{d\varphi}{dt} \cong \frac{\Delta\varphi_i}{\Delta t_i}\ \text{and}\ \frac{d\lambda}{dt} \cong \frac{\Delta\lambda_i}{\Delta t_i}$$

Eq 100. Length of curve:

$$\left[\frac{d\varphi}{dt} \cong \frac{\Delta\varphi_i}{\Delta t_i} \text{ and } \frac{d\lambda}{dt} \cong \frac{\Delta\lambda_i}{\Delta t_i}\right] \Rightarrow$$

$$L_c = \int_a^b \sqrt{E(\varphi)\left(\frac{d\varphi}{dt}\right)^2 + G(\varphi)\left(\frac{d\lambda}{dt}\right)^2}\, dt \cong$$

$$\lim_{\substack{\Delta t \to 0 \\ n \to \infty}} \left(\sum_{i=0}^{n} \left(\left(M\left(\frac{\varphi_i + \varphi_{i-1}}{2}\right)\right) \Delta\varphi_i^2 + \left(\rho\left(\frac{\varphi_i + \varphi_{i-1}}{2}\right)\right) \Delta\lambda_i^2 \right)^{1/2} \right)$$

In general, both φ and λ will most often both vary simultaneously. In some of the examples below for lines of latitude or lines of longitude, allows a simplified length integral (**L_c**), by using deltas for either longitude or latitude, but not both. Which changes the square root of the sum of squares, to a single E or G summation term.

It should be noted that both *E(φ)* and *G(φ)* are always only a function of latitude and always positive (always a sum of squares). In the L_c integrals, the *Δφ* and *Δλ* are squared and then square-rooted, meaning they are also always contributing positively in the numeric integration for the line-length integrals.

Eq 101. N-S Distance

$$L_c(\varphi) = \int_a^b \sqrt{E(\varphi)}\, d\varphi = \lim_{\substack{\Delta t \to 0 \\ n \to \infty}} \left(\sum_{i=0}^{n} \left(\frac{\sqrt{E(\varphi_i)} + \sqrt{E(\varphi_{i-1})}}{2} |\Delta\varphi_i| \right) \right)$$

$$= \int_a^b M(\varphi)\, d\varphi = \lim_{\substack{\Delta t \to 0 \\ n \to \infty}} \left(\sum_{i=0}^{n} \left(\frac{M(\varphi_i) + M(\varphi_{i-1})}{2} |\Delta\varphi_i| \right) \right);$$

$$\Delta\varphi_i = \varphi_i - \varphi_{i-1}$$

Eq 102. E-W Distance

$$L_c(\lambda) = \int_a^b \sqrt{G(\varphi)}\, d\lambda = \lim_{\substack{\Delta t \to 0 \\ n \to \infty}} \left(\sum_{i=0}^{n} \left(\frac{\sqrt{G(\varphi_i)} + \sqrt{G(\varphi_{i-1})}}{2} |\Delta\lambda_i| \right) \right)$$

$$= \int_a^b \rho(\varphi)\, d\lambda = \lim_{\substack{\Delta t \to 0 \\ n \to \infty}} \left(\sum_{i=0}^{n} \left(\frac{\rho(\varphi_i) + \rho(\varphi_{i-1})}{2} |\Delta\lambda_i| \right) \right);$$

$$\Delta\lambda_i = \lambda_i - \lambda_{i-1}$$

It is the case that these two functions are equivalent to local square of the radius of curvature or lines of varying latitude lines (meridians) for $\sqrt{E(\varphi)} = M(\varphi) = a(1-e^2)(1-e^2 \sin^2\varphi)^{-3/2}$ and for lines of varying longitude (parallels) for $\sqrt{G(\varphi)}$ for$\rho(\varphi) = N(\varphi)|\cos\varphi|$.

Annex C Examples

The two tables below describe the length of the meridians (north-south lines) and parallels (east-west) in terms of Δφ and Δλ converted to meters based on the radius of curvature of these curves on the ellipsoid. With this information and the integrals in Annex B. What they show is that using Δφ and Δλ of approximately a degree or less, local distances, and thereby local areas, can be calculated down to the level of centimeter accuracy.

C.1 Length of a Degree of Longitude

The table below uses the equations in Annex B to calculate the length of a degree of longitude along parallels of the various latitudes, partially repeated below the table. This table would be north-south symmetric, so that the value for "φ" is the same as "-φ", The given the radius "r" derives from the computation below the table, circumference is "2πr". The formula below is valid for any φ in radians (2π radians =360°). The radius "ρ" falls out of the calculations in equation below. The radius of curvature in meters at latitude "φ" in the plane of the parallel of latitude.

Eq 103. Parallel radius $$\rho(\varphi) = \sqrt{X^2 + Y^2} = \frac{a\cos\varphi}{\sqrt{1 - e^2\sin^2\varphi}} = N(\varphi)\cos\varphi$$

A full circle is 360° degrees, and 2π radians. The radius of a circle of latitude is ρ(φ) and so the circumference of the circle of parallel is 2πρ(φ). Table 1 show the radius at each latitude (the equator radius is the semimajor axis "a=6,378,137 meters" which is 111.319 km per degree, and 6,378,137 meters per radian)" to the pole where the radius is "0.0". Bomford [3] calls ρ(φ) the radius of the parallel of latitude. The local circumference at latitude "φ" is "$c(\varphi) = 2\pi\rho(\varphi)$". This table did not require any integral but is directly observed by analysis of the ellipsoid, see equation (80). Clynch [9] also deals with the value of the radius function "ρ(φ)". Unlike latitude the length of Δλ along a parallel is constant on any ellipsoid. In each row below the length of the parallel is calculated from the latitude, where the radius of the circular parallel.

Table 1. Length of Parallel of Longitude at each quarter Latitude:

Latitude in Degrees	Latitude in Radians	Radius in Meters of Parallel $\rho(\varphi)$	Circumference in Km	Difference of Parallels in Km	Km in a λ Degree
0.00	0.000000000	6,378,137.000000	40,075.016686	0.000000000	111.31949079
0.25	0.004363323	6,378,076.691178	40,074.637754	0.378931504	111.31843821
0.50	0.008726646	6,377,895.765791	40,073.50966	1.136787733	111.31528046
0.75	0.013089969	6,377,594.227076	40,071.606343	1.894623623	111.31001762
1.00	0.017453293	6,377,172.080428	40,068.953917	2.652425615	111.30264977
1.25	0.021816616	6,376,629.333401	40,065.543737	3.410180150	111.29317705
1.50	0.026179939	6,375,965.995704	40,061.375863	4.167873669	111.28159962
1.75	0.030543262	6,375,182.079207	40,056.450371	4.925492616	111.26791770
2.00	0.034906585	6,374,277.597936	40,050.767347	5.683023432	111.25213152
2.25	0.039269908	6,373,252.568076	40,044.326895	6.440452561	111.23424137
2.50	0.043633231	6,372,107.007966	40,037.129128	7.197766447	111.21424800
2.75	0.047996554	6,370,840.938107	40,029.174177	7.954951535	111.19215049
3.00	0.052359878	6,369,454.381155	40,020.462182	8.711994272	111.16795051
3.25	0.056723201	6,367,947.361922	40,010.993301	9.468881105	111.14164806
3.50	0.061086524	6,366,319.907377	40,000.767703	10.22559848	111.11324362
3.75	0.065449847	6,364,572.046647	39,989.785570	10.98213286	111.08273769
4.00	0.06981317	6,362,703.811015	39,978.047099	11.73847068	111.05013083
4.25	0.074176493	6,360,715.233918	39,965.552501	12.49459840	111.01542361
4.50	0.078539816	6,358,606.350952	39,952.301998	13.2550247	110.97861666
4.75	0.082903139	6,356,377.199865	39,938.295829	14.00616936	110.93971064
5.00	0.087266463	6,354,027.820562	39,923.534244	14.76158551	110.89870623
5.25	0.091629786	6,351,558.255103	39,908.017506	15.51673740	110.85560418
5.50	0.095993109	6,348,968.547703	39,891.745895	16.27161149	110.81040526
5.75	0.100356432	6,346,258.744728	39,874.719700	17.02619423	110.76311028
6.00	0.104719755	6,343,428.894702	39,856.939228	17.78047211	110.71372008
6.25	0.109083078	6,340,479.048298	39,838.404797	18.53443158	110.66223555
6.50	0.113446401	6,337,409.258345	39,819.116738	19.28805913	110.60865760
6.75	0.117809725	6,334,219.579823	39,799.075396	20.04134122	110.55298721
7.00	0.122173048	6,330,910.069865	39,778.281132	20.79426434	110.49522537
7.25	0.126536371	6,327,480.787754	39,756.734317	21.54681497	110.43537310
7.50	0.130899694	6,323,931.794925	39,734.435337	22.29897960	110.37343149
7.75	0.135263017	6,320,263.154963	39,711.384593	23.05074471	110.30940165
8.00	0.13962634	6,316,474.933602	39,687.582496	23.80209679	110.24328471
8.25	0.143989663	6,312,567.198729	39,663.029474	24.55302234	110.17508187
8.50	0.148352986	6,308,540.020374	39,637.725966	25.30350786	110.10479435
8.75	0.152716310	6,304,393.470721	39,611.672426	26.05353986	110.03242341
9.00	0.157079633	6,300,127.624097	39,584.869321	26.80310483	109.95797034
9.25	0.161442956	6,295,742.556979	39,557.317132	27.55218929	109.88143648
9.50	0.165806279	6,291,238.347988	39,529.016352	28.30077975	109.80282320

Latitude in Degrees	Latitude in Radians	Radius in Meters of Parallel ρ(φ)	Circumference in Km	Difference of Parallels in Km	Km in a λ Degree
9.75	0.170169602	6,286,615.077892	39,499.967489	29.04886274	109.72213191
10.00	0.174532925	6,281,872.829603	39,470.171065	29.79642477	109.63936407
10.25	0.178896248	6,277,011.688179	39,439.627612	30.54345238	109.55452114
10.50	0.183259571	6,272,031.740818	39,408.337680	31.28993209	109.46760467
10.75	0.187622895	6,266,933.076864	39,376.301830	32.03585044	109.37861619
11.00	0.191986218	6,261,715.787801	39,343.520636	32.78119398	109.28755732
11.25	0.196349541	6,256,379.967255	39,309.994686	33.52594926	109.19442968
11.50	0.200712864	6,250,925.710992	39,275.724584	34.27010282	109.09923495
11.75	0.205076187	6,245,353.116916	39,240.710942	35.01364122	109.00197484
12.00	0.209439510	6,239,662.285072	39,204.954391	35.75655103	108.90265109
12.25	0.213802833	6,233,853.317642	39,168.455573	36.49881881	108.80126548
12.50	0.218166156	6,227,926.318942	39,131.215141	37.24043114	108.69781984
12.75	0.222529480	6,221,881.395428	39,093.233767	37.98137461	108.59231602
13.00	0.226892803	6,215,718.655689	39,054.512131	38.72163578	108.48475592
13.25	0.231256126	6,209,438.210446	39,015.050930	39.46120127	108.37514147
13.50	0.235619449	6,203,040.172557	38,974.850872	40.20005766	108.26347464
13.75	0.239982772	6,196,524.657008	38,933.912680	40.93819157	108.14975745
14.00	0.244346095	6,189,891.780918	38,892.237091	41.67558959	108.03399192
14.25	0.248709418	6,183,141.663535	38,849.824853	42.41223836	107.91618015
14.50	0.253072742	6,176,274.426237	38,806.676728	43.14812449	107.79632424
14.75	0.257436065	6,169,290.192528	38,762.793493	43.88323462	107.67442637
15.00	0.261799388	6,162,189.088038	38,718.175938	44.61755539	107.55048872
15.25	0.266162711	6,154,971.240525	38,672.824865	45.35107344	107.42451351
15.50	0.270526034	6,147,636.779869	38,626.741089	46.08377543	107.29650303
15.75	0.274889357	6,140,185.838073	38,579.925441	46.81564802	107.16645956
16.00	0.279252680	6,132,618.549263	38,532.378763	47.54667787	107.03438545
16.25	0.283616003	6,124,935.049683	38,484.101912	48.27685166	106.90028309
16.50	0.287979327	6,117,135.477700	38,435.095756	49.00615609	106.76415488
16.75	0.292342650	6,109,219.973795	38,385.361178	49.73457783	106.62600327
17.00	0.296705973	6,101,188.680568	38,334.899074	50.46210360	106.48583076
17.25	0.301069296	6,093,041.742734	38,283.710354	51.18872010	106.34363987
17.50	0.305432619	6,084,779.307120	38,231.795940	51.91441405	106.19943317
17.75	0.309795942	6,076,401.522668	38,179.156768	52.63917218	106.05321324
18.00	0.314159265	6,067,908.540429	38,125.793787	53.36298122	105.90498274
18.25	0.318522588	6,059,300.513566	38,071.707959	54.08582791	105.75474433
18.50	0.322885912	6,050,577.597346	38,016.900260	54.80769902	105.60250072
18.75	0.327249235	6,041,739.949148	37,961.371678	55.52858131	105.44825466
19.00	0.331612558	6,032,787.728451	37,905.123217	56.24846155	105.29200894
19.25	0.335975881	6,023,721.096841	37,848.155890	56.96732652	105.13376636
19.50	0.340339204	6,014,540.218004	37,790.470727	57.68516302	104.97352980

Latitude in Degrees	Latitude in Radians	Radius in Meters of Parallel $\rho(\varphi)$	Circumference in Km	Difference of Parallels in Km	Km in a λ Degree
19.75	0.344702527	6,005,245.257727	37,732.068769	58.40195784	104.81130214
20.00	0.349065850	5,995,836.383896	37,672.951072	59.11769781	104.64708631
20.25	0.353429174	5,986,313.766495	37,613.118702	59.83236974	104.48088528
20.50	0.357792497	5,976,677.577600	37,552.572741	60.54596048	104.31270206
20.75	0.362155820	5,966,927.991385	37,491.314284	61.25845686	104.14253968
21.00	0.366519143	5,957,065.184112	37,429.344439	61.96984574	103.97040122
21.25	0.370882466	5,947,089.334135	37,366.664325	62.68011400	103.79628979
21.50	0.375245789	5,937,000.621898	37,303.275076	63.38924850	103.62020855
21.75	0.379609112	5,926,799.229927	37,239.177840	64.09723614	103.44216067
22.00	0.383972435	5,916,485.342837	37,174.373776	64.80406382	103.26214938
22.25	0.388335759	5,906,059.147324	37,108.864058	65.50971846	103.08017794
22.50	0.392699082	5,895,520.832164	37,042.649871	66.21418697	102.89624964
22.75	0.397062405	5,884,870.588214	36,975.732415	66.91745631	102.71036782
23.00	0.401425728	5,874,108.608405	36,908.112901	67.61951341	102.52253584
23.25	0.405789051	5,863,235.087746	36,839.792556	68.32034524	102.33275710
23.50	0.410152374	5,852,250.223317	36,770.772617	69.01993878	102.14103505
23.75	0.414515697	5,841,154.214269	36,701.054336	69.71828102	101.94737316
24.00	0.418879020	5,829,947.261822	36,630.638977	70.41535895	101.75177494
24.25	0.423242344	5,818,629.569262	36,559.527818	71.11115960	101.55424394
24.50	0.427605667	5,807,201.341941	36,487.722148	71.80566999	101.35478374
24.75	0.431968990	5,795,662.787271	36,415.223270	72.49887717	101.15339797
25.00	0.436332313	5,784,014.114726	36,342.032502	73.19076819	100.95009028
25.25	0.440695636	5,772,255.535836	36,268.151172	73.88133012	100.74486437
25.50	0.445058959	5,760,387.264187	36,193.580622	74.57055004	100.53772395
25.75	0.449422282	5,748,409.515419	36,118.322207	75.25841507	100.32867280
26.00	0.453785606	5,736,322.507223	36,042.377295	75.94491231	100.11771471
26.25	0.458148929	5,724,126.459335	35,965.747266	76.63002889	99.90485352
26.50	0.462512252	5,711,821.593542	35,888.433514	77.31375196	99.69009309
26.75	0.466875575	5,699,408.133671	35,810.437445	77.99606868	99.47343735
27.00	0.471238898	5,686,886.305590	35,731.760479	78.67696622	99.25489022
27.25	0.475602221	5,674,256.337207	35,652.404047	79.35643177	99.03445569
27.50	0.479965544	5,661,518.458466	35,572.369595	80.03445255	98.81213776
27.75	0.484328867	5,648,672.901344	35,491.658579	80.71101577	98.58794050
28.00	0.488692191	5,635,719.899847	35,410.272470	81.38610869	98.36186797
28.25	0.493055514	5,622,659.690012	35,328.212752	82.05971855	98.13392431
28.50	0.497418837	5,609,492.509899	35,245.480919	82.73183262	97.90411366
28.75	0.501782160	5,596,218.599591	35,162.078481	83.40243821	97.67244022
29.00	0.506145483	5,582,838.201193	35,078.006958	84.07152262	97.43890822
29.25	0.510508806	5,569,351.558823	34,993.267885	84.73907318	97.20352190
29.50	0.514872129	5,555,758.918618	34,907.862808	85.40507722	96.96628558

Latitude in Degrees	Latitude in Radians	Radius in Meters of Parallel $\rho(\varphi)$	Circumference in Km	Difference of Parallels in Km	Km in a λ Degree
29.75	0.519235452	5,542,060.528723	34,821.793286	86.06952212	96.72720357
30.00	0.523598776	5,528,256.639293	34,735.060890	86.73239525	96.48628025
30.25	0.527962099	5,514,347.502487	34,647.667206	87.39368402	96.24352002
30.50	0.532325422	5,500,333.372467	34,559.613830	88.05337583	95.99892731
30.75	0.536688745	5,486,214.505397	34,470.902372	88.71145813	95.75250659
31.00	0.541052068	5,471,991.159433	34,381.534454	89.36791838	95.50426237
31.25	0.545415391	5,457,663.594727	34,291.511710	90.02274404	95.25419919
31.50	0.549778714	5,443,232.073422	34,200.835787	90.67592263	95.00232163
31.75	0.554142038	5,428,696.859646	34,109.508346	91.32744164	94.74863429
32.00	0.558505361	5,414,058.219511	34,017.531057	91.97728861	94.49314183
32.25	0.562868684	5,399,316.421111	33,924.905606	92.62545111	94.23584891
32.50	0.567232007	5,384,471.734515	33,831.633689	93.27191671	93.97676025
32.75	0.571595330	5,369,524.431769	33,737.717016	93.91667300	93.71588060
33.00	0.575958653	5,354,474.786886	33,643.157309	94.55970761	93.45321475
33.25	0.580321976	5,339,323.075848	33,547.956300	95.20100817	93.18876750
33.50	0.584685299	5,324,069.576601	33,452.115738	95.84056235	92.92254372
33.75	0.589048623	5,308,714.569050	33,355.637380	96.47835783	92.65454828
34.00	0.593411946	5,293,258.335058	33,258.522998	97.11438232	92.38478611
34.25	0.597775269	5,277,701.158440	33,160.774374	97.74862355	92.11326215
34.50	0.602138592	5,262,043.324960	33,062.393305	98.38106926	91.83998140
34.75	0.606501915	5,246,285.122330	32,963.381598	99.01170724	91.56494888
35.00	0.610865238	5,230,426.840200	32,863.741073	99.64052527	91.28816965
35.25	0.615228561	5,214,468.770164	32,763.473561	100.2675112	91.00964878
35.50	0.619591884	5,198,411.205744	32,662.580909	100.8926528	90.72939141
35.75	0.623955208	5,182,254.442399	32,561.064971	101.5159381	90.44740270
36.00	0.628318531	5,165,998.777511	32,458.927616	102.1373548	90.16368782
36.25	0.632681854	5,149,644.510385	32,356.170725	102.7568909	89.87825201
36.50	0.637045177	5,133,191.942247	32,252.796190	103.3745344	89.59110053
36.75	0.641408500	5,116,641.376236	32,148.805917	103.9902732	89.30223866
37.00	0.645771823	5,099,993.117404	32,044.201822	104.6040953	89.01167173
37.25	0.650135146	5,083,247.472708	31,938.985833	105.2159887	88.71940509
37.50	0.654498469	5,066,404.751009	31,833.159892	105.8259415	88.42544414
37.75	0.658861793	5,049,465.263064	31,726.725950	106.4339418	88.12979431
38.00	0.663225116	5,032,429.321529	31,619.685972	107.0399775	87.83246103
38.25	0.667588439	5,015,297.240946	31,512.041935	107.6440370	87.53344982
38.50	0.671951762	4,998,069.337744	31,403.795827	108.2461083	87.23276619
38.75	0.676315085	4,980,745.930233	31,294.949648	108.8461795	86.93041569
39.00	0.680678408	4,963,327.338603	31,185.505409	109.4442309	86.62640391
39.25	0.685041731	4,945,813.884912	31,075.465134	110.0402749	86.32073648
39.50	0.689405055	4,928,205.893089	30,964.830858	110.6342755	86.01341905

Latitude in Degrees	Latitude in Radians	Radius in Meters of Parallel ρ(φ)	Circumference in Km	Difference of Parallels in Km	Km in a λ Degree
39.75	0.693768378	4,910,503.688925	30,853.604629	111.2262291	85.70445730
40.00	0.698131701	4,892,707.600073	30,741.788505	111.8161240	85.39385696
40.25	0.702495024	4,874,817.956036	30,629.384557	112.4039486	85.08162377
40.50	0.706858347	4,856,835.088170	30,516.394865	112.9896912	84.76776351
40.75	0.711221670	4,838,759.329674	30,402.821525	113.5733402	84.45228201
41.00	0.715584993	4,820,591.015588	30,288.666641	114.1548841	84.13518511
41.25	0.719948316	4,802,330.482789	30,173.932330	114.7343114	83.81647869
41.50	0.724311640	4,783,978.069981	30,058.620719	115.3116105	83.49616866
41.75	0.728674963	4,765,534.117696	29,942.733949	115.8867700	83.17426097
42.00	0.733038286	4,746,998.968287	29,826.274171	116.4597784	82.85076159
42.25	0.737401609	4,728,372.965922	29,709.243546	117.0306244	82.52567652
42.50	0.741764932	4,709,656.456578	29,591.644250	117.5992965	82.19901181
42.75	0.746128255	4,690,849.788040	29,473.478466	118.1657834	81.87077352
43.00	0.750491578	4,671,953.309892	29,354.748393	118.7300739	81.54096776
43.25	0.754854901	4,652,967.373514	29,235.456236	119.2921565	81.20960066
43.50	0.759218225	4,633,892.332076	29,115.604216	119.8520201	80.87667838
43.75	0.763581548	4,614,728.540531	28,995.194562	120.4096535	80.54220712
44.00	0.767944871	4,595,476.355612	28,874.229517	120.9650454	80.20619310
44.25	0.772308194	4,576,136.135828	28,752.711332	121.5181848	79.86864259
44.50	0.776671517	4,556,708.241454	28,630.642272	122.0690605	79.52956187
44.75	0.781034840	4,537,193.034529	28,508.024610	122.6176614	79.18895725
45.00	0.785398163	4,517,590.878849	28,384.860634	123.1639766	78.84683509
45.25	0.789761487	4,497,902.139962	28,261.152639	123.7079949	78.50320177
45.50	0.794124810	4,478,127.185163	28,136.902934	124.2497054	78.15806370
45.75	0.798488133	4,458,266.383487	28,012.113836	124.7890973	77.81142732
46.00	0.802851456	4,438,320.105703	27,886.787677	125.3261595	77.46329910
46.25	0.807214779	4,418,288.724311	27,760.926795	125.8608812	77.11368554
46.50	0.811578102	4,398,172.613532	27,634.533544	126.3932517	76.76259318
46.75	0.815941425	4,377,972.149305	27,507.610284	126.9232600	76.41002857
47.00	0.820304748	4,357,687.709282	27,380.159388	127.4508955	76.05599830
47.25	0.824668072	4,337,319.672818	27,252.183241	127.9761474	75.70050900
47.50	0.829031395	4,316,868.420969	27,123.684236	128.4990051	75.34356732
47.75	0.833394718	4,296,334.336484	26,994.664778	129.0194579	74.98517994
48.00	0.837758041	4,275,717.803798	26,865.127282	129.5374953	74.62535356
48.25	0.842121364	4,255,019.209028	26,735.074176	130.0531065	74.26409493
48.50	0.846484687	4,234,238.939967	26,604.507895	130.5662812	73.90141082
48.75	0.85084801	4,213,377.386073	26,473.430886	131.0770089	73.53730802
49.00	0.855211333	4,192,434.938469	26,341.845607	131.5852791	73.17179335
49.25	0.859574657	4,171,411.989933	26,209.754525	132.0910814	72.80487368
49.50	0.86393798	4,150,308.934891	26,077.160120	132.5944054	72.43655589

Latitude in Degrees	Latitude in Radians	Radius in Meters of Parallel $\rho(\varphi)$	Circumference in Km	Difference of Parallels in Km	Km in a λ Degree
49.75	0.868301303	4,129,126.169414	25,944.064879	133.0952408	72.06684689
50.00	0.872664626	4,107,864.091207	25,810.471302	133.5935774	71.69575362
50.25	0.877027949	4,086,523.099606	25,676.381897	134.0894049	71.32328305
50.50	0.881391272	4,065,103.595569	25,541.799184	134.5827131	70.94944218
50.75	0.885754595	4,043,605.981672	25,406.725692	135.0734918	70.57423803
51.00	0.890117919	4,022,030.662098	25,271.163961	135.5617309	70.19767767
51.25	0.894481242	4,000,378.042635	25,135.116541	136.0474205	69.81976817
51.50	0.898844565	3,978,648.530665	24,998.585990	136.5305503	69.44051664
51.75	0.903207888	3,956,842.535161	24,861.574880	137.0111106	69.05993022
52.00	0.907571211	3,934,960.466675	24,724.085789	137.4890912	68.67801608
52.25	0.911934534	3,913,002.737339	24,586.121306	137.9644823	68.29478141
52.50	0.916297857	3,890,969.760847	24,447.684032	138.4372742	67.91023342
52.75	0.920661180	3,868,861.952457	24,308.776575	138.9074568	67.52437938
53.00	0.925024504	3,846,679.728981	24,169.401555	139.3750206	67.13722654
53.25	0.929387827	3,824,423.508777	24,029.561599	139.8399558	66.74878222
53.50	0.933751150	3,802,093.711742	23,889.259346	140.3022526	66.35905374
53.75	0.938114473	3,779,690.759303	23,748.497445	140.7619016	65.96804846
54.00	0.942477796	3,757,215.074415	23,607.278551	141.2188931	65.57577375
54.25	0.946841119	3,734,667.081548	23,465.605334	141.6732175	65.18223704
54.50	0.951204442	3,712,047.206682	23,323.480469	142.1248654	64.78744575
54.75	0.955567765	3,689,355.877298	23,180.906641	142.5738274	64.39140734
55.00	0.959931089	3,666,593.522374	23,037.886547	143.0200940	63.99412930
55.25	0.964294412	3,643,760.572373	22,894.422891	143.4636560	63.59561914
55.50	0.968657735	3,620,857.459237	22,750.518387	143.9045039	63.19588441
55.75	0.973021058	3,597,884.616380	22,606.175759	144.3426287	62.79493266
56.00	0.977384381	3,574,842.478680	22,461.397738	144.7780210	62.39277149
56.25	0.981747704	3,551,731.482471	22,316.187066	145.2106718	61.98940852
56.50	0.986111027	3,528,552.065534	22,170.546494	145.6405719	61.58485137
56.75	0.990474351	3,505,304.667091	22,024.478781	146.0677123	61.17910773
57.00	0.994837674	3,481,989.727795	21,877.986697	146.4920840	60.77218527
57.25	0.999200997	3,458,607.689725	21,731.073019	146.9136781	60.36409172
57.50	1.003564320	3,435,158.996373	21,583.740534	147.3324855	59.95483482
57.75	1.007927643	3,411,644.092642	21,435.992036	147.7484976	59.54442232
58.00	1.012290966	3,388,063.424832	21,287.830331	148.1617055	59.13286203
58.25	1.016654289	3,364,417.440635	21,139.258230	148.5721005	58.72016175
58.50	1.021017612	3,340,706.589127	20,990.278556	148.9796738	58.30632932
58.75	1.025380936	3,316,931.320758	20,840.894140	149.3844169	57.89137261
59.00	1.029744259	3,293,092.087345	20,691.107818	149.7863211	57.47529950
59.25	1.034107582	3,269,189.342061	20,540.922440	150.1853780	57.05811789
59.50	1.038470905	3,245,223.539431	20,390.340861	150.5815790	56.63983573

Latitude in Degrees	Latitude in Radians	Radius in Meters of Parallel $\rho(\varphi)$	Circumference in Km	Difference of Parallels in Km	Km in a λ Degree
59.75	1.042834228	3,221,195.135320	20,239.365946	150.9749157	56.22046096
60.00	1.047197551	3,197,104.586924	20,088.00566	151.3653797	55.80000157
60.25	1.051560874	3,172,952.352764	19,936.247603	151.7529628	55.37846556
60.50	1.055924197	3,148,738.892675	19,784.109947	152.1376567	54.95586096
60.75	1.060287521	3,124,464.667800	19,631.590494	152.5194531	54.53219582
61.00	1.064650844	3,100,130.140577	19,478.692150	152.8983439	54.10747819
61.25	1.069014167	3,075,735.774734	19,325.417829	153.2743210	53.68171619
61.50	1.073377490	3,051,282.035278	19,171.770452	153.6473765	53.25491792
61.75	1.077740813	3,026,769.388488	19,017.752950	154.0175021	52.82709153
62.00	1.082104136	3,002,198.301904	18,863.368260	154.3846902	52.39824517
62.25	1.086467459	2,977,569.244319	18,708.619327	154.7489328	51.96838702
62.50	1.090830782	2,952,882.685768	18,553.509105	155.1102220	51.53752529
62.75	1.095194106	2,928,139.097524	18,398.040555	155.4685501	51.10566821
63.00	1.099557429	2,903,338.952081	18,242.216645	155.8239095	50.67282402
63.25	1.103920752	2,878,482.723153	18,086.040353	156.1762924	50.23900098
63.50	1.108284075	2,853,570.885660	17,929.514662	156.5256913	49.80420739
63.75	1.112647398	2,828,603.915717	17,772.642563	156.8720987	49.36845156
64.00	1.117010721	2,803,582.290630	17,615.427056	157.2155071	48.93174182
64.25	1.121374044	2,778,506.488883	17,457.871147	157.5559091	48.49408652
64.50	1.125737368	2,753,376.990129	17,299.977850	157.8932973	48.05549403
64.75	1.130100691	2,728,194.275181	17,141.750185	158.2276646	47.61597274
65.00	1.134464014	2,702,958.826003	16,983.191181	158.5590035	47.17553106
65.25	1.138827337	2,677,671.125698	16,824.303874	158.8873070	46.73417743
65.50	1.143190660	2,652,331.658502	16,665.091306	159.2125680	46.29192030
65.75	1.147553983	2,626,940.909770	16,505.556527	159.5347794	45.84876813
66.00	1.151917306	2,601,499.365971	16,345.702593	159.8539342	45.40472942
66.25	1.156280629	2,576,007.514674	16,185.532567	160.1700255	44.95981269
66.50	1.160643953	2,550,465.844542	16,025.049521	160.4830465	44.51402645
66.75	1.165007276	2,524,874.845319	15,864.256531	160.7929903	44.06737925
67.00	1.169370599	2,499,235.007819	15,703.156680	161.0998503	43.61987967
67.25	1.173733922	2,473,546.823924	15,541.753061	161.4036196	43.17153628
67.50	1.178097245	2,447,810.786562	15,380.048769	161.7042918	42.72235769
67.75	1.182460568	2,422,027.389707	15,218.046909	162.0018603	42.27235252
68.00	1.186823891	2,396,197.128365	15,055.750590	162.2963185	41.82152942
68.25	1.191187214	2,370,320.498563	14,893.162930	162.5876602	41.36989703
68.50	1.195550538	2,344,397.997340	14,730.287051	162.8758788	40.91746403
68.75	1.199913861	2,318,430.122737	14,567.126083	163.1609682	40.46423912
69.00	1.204277184	2,292,417.373786	14,403.683161	163.4429220	40.01023100
69.25	1.208640507	2,266,360.25502	14,239.961427	163.7217342	39.55544841
69.50	1.213003830	2,240,259.253868	14,075.964028	163.9973986	39.09990008

Latitude in Degrees	Latitude in Radians	Radius in Meters of Parallel $\rho(\varphi)$	Circumference in Km	Difference of Parallels in Km	Km in a λ Degree
69.75	1.217367153	2,214,114.885828	13,911.694119	164.2699091	38.64359478
70.00	1.221730476	2,187,927.649279	13,747.154859	164.5392599	38.18654128
70.25	1.226093800	2,161,698.048054	13,582.349414	164.8054450	37.72874837
70.50	1.230457123	2,135,426.586917	13,417.280955	165.0684586	37.27022488
70.75	1.234820446	2,109,113.771550	13,251.952661	165.3282949	36.81097961
71.00	1.239183769	2,082,760.108543	13,086.367712	165.5849482	36.35102142
71.25	1.243547092	2,056,366.105383	12,920.529300	165.8384129	35.89035917
71.50	1.247910415	2,029,932.270445	12,754.440616	166.0886833	35.42900171
71.75	1.252273738	2,003,459.112979	12,588.104862	166.3357540	34.96695795
72.00	1.256637061	1,976,947.143101	12,421.525243	166.5796196	34.50423678
72.25	1.261000385	1,950,396.871779	12,254.704968	166.8202747	34.04084713
72.50	1.265363708	1,923,808.810830	12,087.647254	167.0577139	33.57679793
72.75	1.269727031	1,897,183.472899	11,920.355322	167.2919321	33.11209812
73.00	1.274090354	1,870,521.371456	11,752.832398	167.5229240	32.64675666
73.25	1.278453677	1,843,823.020780	11,585.081713	167.7506847	32.18078254
73.50	1.282817000	1,817,088.935952	11,417.106504	167.9752090	31.71418473
73.75	1.287180323	1,790,319.632843	11,248.910012	168.1964920	31.24697226
74.00	1.291543646	1,763,515.628099	11,080.495483	168.4145288	30.77915412
74.25	1.295906970	1,736,677.439137	10,911.866169	168.6293146	30.31073936
74.50	1.300270293	1,709,805.584127	10,743.025324	168.8408446	29.84173701
74.75	1.304633616	1,682,900.581986	10,573.976210	169.0491141	29.37215614
75.00	1.308996939	1,655,962.952365	10,404.722092	169.2541186	28.90200581
75.25	1.313360262	1,628,993.215636	10,235.266238	169.4558536	28.43129511
75.50	1.317723585	1,601,991.892885	10,065.611924	169.6543144	27.96003312
75.75	1.322086908	1,574,959.505896	9,895.762427	169.8494967	27.48822896
76.00	1.326450232	1,547,896.577144	9,725.721031	170.0413963	27.01589175
76.25	1.330813555	1,520,803.629781	9,555.491022	170.2300088	26.54303062
76.50	1.335176878	1,493,681.187625	9,385.075692	170.4153301	26.06965470
76.75	1.339540201	1,466,529.775149	9,214.478336	170.5973559	25.59577315
77.00	1.343903524	1,439,349.917470	9,043.702253	170.7760824	25.12139515
77.25	1.348266847	1,412,142.140339	8,872.750748	170.9515055	24.64652986
77.50	1.352630170	1,384,906.970125	8,701.627127	171.1236213	24.17118646
77.75	1.356993493	1,357,644.933808	8,530.334700	171.2924260	23.69537417
78.00	1.361356817	1,330,356.558966	8,358.876785	171.4579159	23.21910218
78.25	1.365720140	1,303,042.373763	8,187.256697	171.6200871	22.74237972
78.50	1.370083463	1,275,702.906938	8,015.477761	171.7789363	22.26521600
78.75	1.374446786	1,248,338.687793	7,843.543302	171.9344597	21.78762028
79.00	1.378810109	1,220,950.246182	7,671.456648	172.0866539	21.30960180
79.25	1.383173432	1,193,538.112498	7,499.221132	172.2355156	20.83116981
79.50	1.387536755	1,166,102.817665	7,326.840091	172.3810414	20.35233359

Latitude in Degrees	Latitude in Radians	Radius in Meters of Parallel ρ(φ)	Circumference in Km	Difference of Parallels in Km	Km in a λ Degree
79.75	1.391900078	1,138,644.893121	7,154.316863	172.5232281	19.87310240
80.00	1.396263402	1,111,164.870810	6,981.654790	172.6620724	19.39348553
80.25	1.400626725	1,083,663.283170	6,808.857219	172.7975714	18.91349227
80.50	1.404990048	1,056,140.663121	6,635.927497	172.9297219	18.43313194
80.75	1.409353371	1,028,597.544051	6,462.868976	173.0585211	17.95241382
81.00	1.413716694	1,001,034.459806	6,289.685010	173.1839659	17.47134725
81.25	1.418080017	973,451.944681	6,116.378956	173.3060538	16.98994154
81.50	1.422443340	945,850.533404	5,942.954174	173.4247818	16.50820604
81.75	1.426806664	918,230.761124	5,769.414027	173.5401474	16.02615007
82.00	1.431169987	890,593.163403	5,595.761879	173.6521479	15.54378300
82.25	1.435533310	862,938.276200	5,422.001098	173.7607809	15.06111416
82.50	1.439896633	835,266.635864	5,248.135054	173.8660440	14.57815293
82.75	1.444259956	807,578.779115	5,074.167119	173.9679347	14.09490866
83.00	1.448623279	779,875.243040	4,900.100668	174.0664508	13.61139075
83.25	1.452986602	752,156.565074	4,725.939078	174.1615901	13.12760855
83.50	1.457349925	724,423.282993	4,551.685728	174.2533505	12.64357147
83.75	1.461713249	696,675.934900	4,377.343998	174.3417298	12.15928888
84.00	1.466076572	668,915.059213	4,202.917272	174.4267262	11.67477020
84.25	1.470439895	641,141.194654	4,028.408934	174.5083377	11.19002482
84.50	1.474803218	613,354.880236	3,853.822372	174.5865625	10.70506214
84.75	1.479166541	585,556.655249	3,679.160973	174.6613988	10.21989159
85.00	1.483529864	557,747.059254	3,504.428128	174.7328450	9.73452258
85.25	1.487893187	529,926.632063	3,329.627228	174.8008994	9.24896452
85.50	1.492256510	502,095.913735	3,154.761668	174.8655605	8.76322686
85.75	1.496619834	474,255.444558	2,979.834841	174.9268269	8.27731900
86.00	1.50983157	446,405.765037	2,804.850144	174.9846972	7.79125040
86.25	1.505346480	418,547.415888	2,629.810974	175.0391701	7.30503048
86.50	1.509709803	390,680.938017	2,454.720730	175.0902443	6.81866869
86.75	1.514073126	362,806.872517	2,279.582811	175.1379188	6.33217447
87.00	1.518436449	334,925.760648	2,104.400618	175.1821924	5.84555727
87.25	1.522799772	307,038.143829	1,929.177554	175.2230642	5.35882654
87.50	1.527163095	279,144.563626	1,753.917021	175.2605333	4.87199172
87.75	1.531526419	251,245.561738	1,578.622422	175.2945987	4.38506228
88.00	1.535889742	223,341.679987	1,403.297162	175.3252598	3.89804767
88.25	1.540253065	195,433.460304	1,227.944646	175.3525159	3.41095735
88.50	1.544616388	167,521.444716	1,052.568280	175.3763662	2.92380078
88.75	1.548979711	139,606.175338	877.171470	175.3968104	2.43658742
89.00	1.553343034	111,688.194356	701.757622	175.4138479	1.94932673
89.25	1.557706357	83,768.044017	526.330143	175.4274784	1.46202818
89.50	1.562069681	55,846.266619	350.892442	175.4377015	0.97470123

Latitude in Degrees	Latitude in Radians	Radius in Meters of Parallel ρ(φ)	Circumference in Km	Difference of Parallels in Km	Km in a λ Degree
89.75	1.566433004	27,923.404494	175.447925	175.4445170	0.48735535
90.00	1.570796327	0.00000	0.00000	175.4479248	0.0000000

This could have been done using a numeric integral, which drops the φ based terms since φ is a constant along a parallel and thereby produces zero to the integral. For example, if Δφ=0, then the $\sqrt{G(\varphi)}$ function is a constant, and the $d\varphi/dt \equiv 0$ which implies

Eq 104.
$$L_c = \int_a^b \sqrt{E(\varphi)\left(\frac{d\varphi}{dt}\right)^2 + G(\varphi)\left(\frac{d\lambda}{dt}\right)^2}\,dt$$

Eq 105.
$$E(\varphi) = \left(M(\varphi)\right)^2$$
$$\sqrt{E(\varphi)} = M(\varphi) = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{-3/2}}$$

$$N(\varphi) = \frac{a}{\sqrt{1-e^2\sin^2\varphi}} \qquad \rho(\varphi) = N(\varphi)\cos\varphi$$

Eq 106.
$$G(\varphi) = N^2(\varphi)\cos^2\varphi = \rho^2(\varphi) = \left(\frac{a^2\cos^2\varphi}{1-e^2\sin^2\varphi}\right)$$
$$\sqrt{G(\varphi)} = N(\varphi)\cos\varphi = \rho(\varphi)$$

C.2 Length of a meridian of north-south latitude

Table 2 below calculates the north-south distances for each quarter degree between 0° (equator) to 90° (either pole). Note that the calculations use radians. The function m(φ) describes the radius of curvature along the vertical meridian arcs. The radius of curvature of an arc at a point is the radius of the best-fitting circle tangent to that arc. See Burkholder [4], Wikipedia [33].

Eq 107.
$$M(\varphi) = a(1-e^2)(1-e^2\sin^2\varphi)^{-3/2}$$
$$m(\varphi_n) = \int_0^{\varphi_n} M(\varphi)d\varphi \cong \sum_{i=1}^{90*4}\left(\frac{M(\varphi_i)-M(\varphi_{i-1})}{2}\right)\Delta\varphi_i$$
$$\Delta\varphi_i = \left|\varphi_i - \varphi_{i-1}\right|$$

The same table can be calculated using the length integral in latitudes along a meridian $\int_0^{\frac{\pi}{2}} \sqrt{E(\varphi)}\, d\varphi = \int_0^{\frac{\pi}{2}} M(\varphi)\, d\varphi$; which is described in equation (81. Using a spread sheet to do the calculations, the difference between using $\sqrt{\boldsymbol{E}}$ and M tracked until the end which showed a 0.13-meter difference, or a little less than 5.1 inches or 13 cm in about 10,001 km. See Latitude, Wikipedia page is https://en.wikipedia.org/wiki/Latitude. Using a different spread sheet with a delta of a tenth of a degree, the difference between $\sqrt{\boldsymbol{E}}$ and M from 0 to 360 degrees was tested; the two were identical to the limits of the application which agreed to 8 decimal places and 7-digit integer parts meaning that the two values are within 10 nanometers, which would probably mean that the spread sheet was using double precision floating numbers, and found no difference in values. The editor is still working on a proof that then two are mathematical identical functions. The columns are:

1. **Degrees:** is the current latitude (φ) in decimal degrees, in 0.25 increments, beginning at the equator and traversing to either pole.
2. **M(φ) in meters:** This holds the *M(φ)*-values for the summation difference between each quarter degree $[M(\varphi_{i+1}) - M(\varphi_i)]$ that need to be multiplied by the *Δφ* to achieve the delta distance for *M(φ)*.
3. **KM up to φ_i:** This holds the i[th] summation term in meters for the numeric integral e.g. $M(\varphi_n) \cong \sum_{i=1}^{n}[M(\varphi_i) + M(\varphi_{i-1})](\Delta\varphi)$. The first value is null, since each sub-section is the interval between two values of latitude, i.e. the Δφ's. This column is the partial sum of the summation, and therefore the current best value for the distance from the equator from 0 to the current φ (i.e. 0° to 0.25°…,90°). The summation in Wikipedia is 10001.965729 km and our value in meters is 10,001,965.7293125 from the Wikipedia page https://en.wikipedia.org/wiki/Latitude, [32].
4. **Delta KM:** is the difference since the last partial sum of the numerical integration, and in the light of column 1, the distance between the current φ and the directly previous. This is the partial sums that create the previous column.
5. **Latitude (Radians):** This column lists the angles in radians (needed in the integral) from 0° to 90° in quarter degrees or approximately 0.0043633231 radians.

The consistency of these values is based on the size of the interval 0.25°, and the accuracy of the implied linearity in the summation approximation of the integral. Note that the various values are consistent in accuracy for the double precision floating point that is used by the spread sheet. The function m(φ) would be accurate up to 17 significant decimal digits. Table 2 can be trusted for about 9 digits and the result for kilometers for φ in [0°,90°], which should be trusted probably to the decimeter, and maybe to the centimeter, so the final distance 10,001.965729313 is probably correct to 6 decimal places in this example, or more precisely for the ellipsoid. Wikipedia (see https://en.wikipedia.org/wiki/Latitude, [32]) gives a value (10,001.965729 km) which agrees with the table's value to the millimeter. This is consistent with the delta kilometers between quarter degrees, which is usually near linear at the sub-meter level, which validates the use of only quarter-degree increments and the linear interpolation for the integral $\Delta\varphi_i = |\varphi_i - \varphi_{i-1}| = 0.25^o$. The numeric integral using M(φ) is in equation Eq 4. More accurate numeric integrals could use a delta of 0.1°; as the limits of double precision numbers is usually set in the 15[th] or 17[th] decimal digit. The table below uses 12 decimal digits. See, Weintrit [36] which states 10,001,965.729 as a best approximation, and references

Bomford, 1985, as one of the best consensus at 10,001,965.72931360 by looking at Weintrit's assessed 7 best approximations with 4 of them range between 10,001,965.7293127 and 10,001,965.7293136 which puts this calculation within a micrometer of the consensus value. As pointed out earlier, the similar values for "λ" is accurate based on the ellipsoid regardless of $\Delta\lambda$. This implies that summations at Δ-angles of a quarter degree 0.25° (0.004363323 radian) or smaller are sufficient to maintain centimeter accuracy or better, using double-precision arithmetic. In the geometries of ISO 19107, most "direct positions" used in coordinate strings will probably much smaller Δ-angles than a quarter degree, which is 27.64km or larger in latitudes near a pole, approaching 27.9235km.

Table 2. Length of Meridian Equator to Pole $\int m(\varphi)d\varphi$ (or $\sqrt{E}$)

Degree φ	M(φ)	KM up to φ	delta KM	Radian=φ
0.00	6,335,439.32729	0	0	0.000000000
0.25	6,335,440.53848	27.643571598	27.64357160	0.0043633231
0.50	6,335,444.17194	55.287153765	27.64358217	0.0087266463
0.75	6,335,450.22742	82.930757070	27.64360331	0.0130899694
1.00	6,335,458.70445	110.574392080	27.64363501	0.0174532925
1.25	6,335,469.60242	138.218069360	27.64367728	0.0218166156
1.50	6,335,482.92051	165.861799471	27.64373011	0.0261799388
1.75	6,335,498.65773	193.505592971	27.64379350	0.0305432619
2.00	6,335,516.81291	221.149460413	27.64386744	0.0349065850
2.25	6,335,537.38471	248.793412344	27.64395193	0.0392699082
2.50	6,335,560.37161	276.437459305	27.64404696	0.0436332313
2.75	6,335,585.77188	304.081611831	27.64415253	0.0479965544
3.00	6,335,613.58365	331.725880447	27.64426862	0.0523598776
3.25	6,335,643.80485	359.370275672	27.64439522	0.0567232007
3.50	6,335,676.43325	387.014808013	27.64453234	0.0610865238
3.75	6,335,711.46640	414.659487969	27.64467996	0.0654498469
4.00	6,335,748.90172	442.304326026	27.64483806	0.0698131701
4.25	6,335,788.73642	469.949332661	27.64500663	0.0741764932
4.50	6,335,830.96755	497.594518335	27.64518567	0.0785398163
4.75	6,335,875.59196	525.239893499	27.64537516	0.0829031395
5.00	6,335,922.60635	552.885468587	27.64557509	0.0872664626
5.25	6,335,972.00723	580.531254021	27.64578543	0.0916297857
5.50	6,336,023.79091	608.177260206	27.64600618	0.0959931089
5.75	6,336,077.95357	635.823497530	27.64623732	0.1003564320
6.00	6,336,134.49117	663.469976364	27.64647883	0.1047197551
6.25	6,336,193.39951	691.116707062	27.64673070	0.1090830782
6.50	6,336,254.67423	718.763699959	27.64699290	0.1134464014
6.75	6,336,318.31076	746.410965370	27.64726541	0.1178097245
7.00	6,336,384.30438	774.058513590	27.64754822	0.1221730476
7.25	6,336,452.65018	801.706354893	27.64784130	0.1265363708
7.50	6,336,523.34309	829.354499531	27.64814464	0.1308996939
7.75	6,336,596.37786	857.002957735	27.64845820	0.1352630170
8.00	6,336,671.74906	884.651739710	27.64878198	0.1396263402

Degree φ	M(φ)	KM up to φ	delta KM	Radian=φ
8.25	6,336,749.45108	912.300855640	27.64911593	0.1439896633
8.50	6,336,829.47815	939.950315681	27.64946004	0.1483529864
8.75	6,336,911.82433	967.600129965	27.64981428	0.1527163095
9.00	6,336,996.48350	995.25308598	27.65017863	0.1570796327
9.25	6,337,083.44935	1,022.900861659	27.65055306	0.1614429558
9.50	6,337,172.71543	1,050.551799199	27.65093754	0.1658062789
9.75	6,337,264.27510	1,078.203131239	27.65133204	0.1701696021
10.00	6,337,358.12155	1,105.854867773	27.65173653	0.1745329252
10.25	6,337,454.24781	1,133.507018763	27.65215099	0.1788962483
10.50	6,337,552.64673	1,161.159594140	27.65257538	0.1832595715
10.75	6,337,653.31098	1,188.812603807	27.65300967	0.1876228946
11.00	6,337,756.23309	1,216.466057630	27.65345382	0.1919862177
11.25	6,337,861.40539	1,244.119965444	27.65390781	0.1963495408
11.50	6,337,968.82007	1,271.774337051	27.65437161	0.2007128640
11.75	6,338,078.46914	1,299.429182218	27.65484517	0.2050761871
12.00	6,338,190.34443	1,327.084510676	27.65532846	0.2094395102
12.25	6,338,304.43763	1,354.740332121	27.65582144	0.2138028334
12.50	6,338,420.74023	1,382.396656212	27.65632409	0.2181661565
12.75	6,338,539.24360	1,410.053492569	27.65683636	0.2225294796
13.00	6,338,659.93891	1,437.710850777	27.65735821	0.2268928028
13.25	6,338,782.81717	1,465.368740381	27.65788960	0.2312561259
13.50	6,338,907.86925	1,493.027170884	27.65843050	0.2356194490
13.75	6,339,035.08582	1,520.686151752	27.65898087	0.2399827721
14.00	6,339,164.45742	1,548.345692409	27.65954066	0.2443460953
14.25	6,339,295.97442	1,576.005802237	27.66010983	0.2487094184
14.50	6,339,429.62702	1,603.666490574	27.66068834	0.2530727415
14.75	6,339,565.40526	1,631.327766719	27.66127614	0.2574360647
15.00	6,339,703.29904	1,658.989639923	27.66187320	0.2617993878
15.25	6,339,843.29809	1,686.652119396	27.66247947	0.2661627109
15.50	6,339,985.39196	1,714.315214300	27.66309490	0.2705260341
15.75	6,340,129.57009	1,741.978933752	27.66371945	0.2748893572
16.00	6,340,275.82171	1,769.643286824	27.66435307	0.2792526803
16.25	6,340,424.13594	1,797.308282539	27.66499571	0.2836160034
16.50	6,340,574.50172	1,824.973929872	27.66564733	0.2879793266
16.75	6,340,726.90783	1,852.640237752	27.66630788	0.2923426497
17.00	6,340,881.34292	1,880.307215055	27.66697730	0.2967059728
17.25	6,341,037.79548	1,907.974870609	27.66765555	0.3010692960
17.50	6,341,196.25383	1,935.643213193	27.66834258	0.3054326191
17.75	6,341,356.70615	1,963.312251532	27.66903834	0.3097959422
18.00	6,341,519.14049	1,990.981994300	27.66974277	0.3141592654
18.25	6,341,683.54472	2,018.652450119	27.67045582	0.3185225885
18.50	6,341,849.90658	2,046.323627558	27.67117744	0.3228859116

Degree φ	M(φ)	KM up to φ	delta KM	Radian=φ
18.75	6,342,018.21365	2,073.995535131	27.67190757	0.3272492347
19.00	6,342,188.45337	2,101.668181299	27.67264617	0.3316125579
19.25	6,342,360.61304	2,129.341574467	27.67339317	0.3359758810
19.50	6,342,534.67980	2,157.015722983	27.67414852	0.3403392041
19.75	6,342,710.64066	2,184.690635141	27.67491216	0.3447025273
20.00	6,342,888.48248	2,212.366319177	27.67568404	0.3490658504
20.25	6,343,068.19198	2,240.042783269	27.67646409	0.3534291735
20.50	6,343,249.75573	2,267.720035537	27.67725227	0.3577924967
20.75	6,343,433.16018	2,295.398084042	27.67804850	0.3621558198
21.00	6,343,618.39162	2,323.076936785	27.67885274	0.3665191429
21.25	6,343,805.43621	2,350.756601709	27.67966492	0.3708824660
21.50	6,343,994.27996	2,378.437086694	27.68048499	0.3752457892
21.75	6,344,184.90878	2,406.118399560	27.68131287	0.3796091123
22.00	6,344,377.30840	2,433.800548064	27.68214850	0.3839724354
22.25	6,344,571.46444	2,461.483539902	27.68299184	0.3883357586
22.50	6,344,767.36237	2,489.167382706	27.68384280	0.3926990817
22.75	6,344,964.98756	2,516.852084044	27.68470134	0.3970624048
23.00	6,345,164.32522	2,544.537651420	27.68556738	0.4014257280
23.25	6,345,365.36042	2,572.224092275	27.68644085	0.4057890511
23.50	6,345,568.07814	2,599.911413982	27.68732171	0.4101523742
23.75	6,345,772.46320	2,627.599623849	27.68820987	0.4145156973
24.00	6,345,978.5029	2,655.288729118	27.68910527	0.4188790205
24.25	6,346,186.17400	2,682.978736965	27.69000785	0.4232423436
24.50	6,346,395.46878	2,710.669654495	27.69091753	0.4276056667
24.75	6,346,606.36895	2,738.361488749	27.69183425	0.4319689899
25.00	6,346,818.85872	2,766.054246697	27.69275795	0.4363323130
25.25	6,347,032.92217	2,793.747935239	27.69368854	0.4406956361
25.50	6,347,248.54325	2,821.442561207	27.69462597	0.4450589593
25.75	6,347,465.70581	2,849.138131363	27.69557016	0.4494222824
26.00	6,347,684.39358	2,876.834652396	27.69652103	0.4537856055
26.25	6,347,904.59016	2,904.532130927	27.69747853	0.4581489286
26.50	6,348,126.27904	2,932.230573502	27.69844258	0.4625122518
26.75	6,348,349.44360	2,959.929986597	27.69941309	0.4668755749
27.00	6,348,574.06710	2,987.630376614	27.70039002	0.4712388980
27.25	6,348,800.13268	3,015.331749882	27.70137327	0.4756022212
27.50	6,349,027.62339	3,043.034112657	27.70236277	0.4799655443
27.75	6,349,256.52215	3,070.737471118	27.70335846	0.4843288674
28.00	6,349,486.81178	3,098.441831374	27.70436026	0.4886921906
28.25	6,349,718.47500	3,126.147199454	27.70536808	0.4930555137
28.50	6,349,951.49441	3,153.853581314	27.70638186	0.4974188368
28.75	6,350,185.85252	3,181.560982834	27.70740152	0.5017821599
29.00	6,350,421.53171	3,209.269409816	27.70842698	0.5061454831

Degree φ	M(φ)	KM up to φ	delta KM	Radian=φ
29.25	6,350,658.51428	3,236.978867986	27.70945817	0.5105088062
29.50	6,350,896.78242	3,264.689362993	27.71049501	0.5148721293
29.75	6,351,136.31824	3,292.400900406	27.71153741	0.5192354525
30.00	6,351,377.10372	3,320.113485717	27.71258531	0.5235987756
30.25	6,351,619.12075	3,347.827124341	27.71363862	0.5279620987
30.50	6,351,862.35115	3,375.541821609	27.71469727	0.5323254219
30.75	6,352,106.77661	3,403.257582778	27.71576117	0.5366887450
31.00	6,352,352.37875	3,430.974413022	27.71683024	0.5410520681
31.25	6,352,599.13909	3,458.692317433	27.71790441	0.5454153912
31.50	6,352,847.03906	3,486.411301026	27.71898359	0.5497787144
31.75	6,353,096.05999	3,514.131368732	27.72006771	0.5541420375
32.00	6,353,346.18315	3,541.852525402	27.72115667	0.5585053606
32.25	6,353,597.38969	3,569.574775803	27.72225040	0.5628686838
32.50	6,353,849.66071	3,597.298124622	27.72334882	0.5672320069
32.75	6,354,102.97718	3,625.022576462	27.72445184	0.5715953300
33.00	6,354,357.32003	3,652.748135843	27.72555938	0.5759586532
33.25	6,354,612.67009	3,680.474807201	27.72667136	0.5803219763
33.50	6,354,869.00812	3,708.202594889	27.72778769	0.5846852994
33.75	6,355,126.31478	3,735.931503177	27.72890829	0.5890486225
34.00	6,355,384.57067	3,763.661536247	27.73003307	0.5934119457
34.25	6,355,643.75632	3,791.392698199	27.73116195	0.5977752688
34.50	6,355,903.85217	3,819.124993048	27.73229485	0.6021385919
34.75	6,356,164.83860	3,846.858424723	27.73343167	0.6065019151
35.00	6,356,426.69592	3,874.592997065	27.73457234	0.6108652382
35.25	6,356,689.40435	3,902.328713832	27.73571677	0.6152285613
35.50	6,356,952.94406	3,930.065578695	27.73686486	0.6195918845
35.75	6,357,217.29515	3,957.803595236	27.73801654	0.6239552076
36.00	6,357,482.43765	3,985.542766954	27.73917172	0.6283185307
36.25	6,357,748.35154	4,013.283097257	27.74033030	0.6326818538
36.50	6,358,015.01671	4,041.024589467	27.74149221	0.6370451770
36.75	6,358,282.41302	4,068.767246818	27.74265735	0.6414085001
37.00	6,358,550.52026	4,096.511072457	27.74382564	0.6457718232
37.25	6,358,819.31814	4,124.256069441	27.74499698	0.6501351464
37.50	6,359,088.78634	4,152.002240740	27.74617130	0.6544984695
37.75	6,359,358.90448	4,179.749589233	27.74734849	0.6588617926
38.00	6,359,629.65213	4,207.498117713	27.74852848	0.6632251158
38.25	6,359,901.00878	4,235.247828881	27.74971117	0.6675884389
38.50	6,360,172.95391	4,262.998725349	27.75089647	0.6719517620
38.75	6,360,445.46692	4,290.750809641	27.75208429	0.6763150851
39.00	6,360,718.52718	4,318.504084188	27.75327455	0.6806784083
39.25	6,360,992.11401	4,346.258551335	27.75446715	0.6850417314
39.50	6,361,266.20668	4,374.014213333	27.75566200	0.6894050545

Degree φ	M(φ)	KM up to φ	delta KM	Radian=φ
39.75	6,361,540.78443	4,401.771072345	27.75685901	0.6937683777
40.00	6,361,815.82643	4,429.529130440	27.75805810	0.6981317008
40.25	6,362,091.31185	4,457.288389601	27.75925916	0.7024950239
40.50	6,362,367.21980	4,485.048851714	27.76046211	0.7068583471
40.75	6,362,643.52934	4,512.810518580	27.76166687	0.7112216702
41.00	6,362,920.21953	4,540.573391904	27.76287332	0.7155849933
41.25	6,363,197.26936	4,568.337473301	27.76408140	0.7199483164
41.50	6,363,474.65782	4,596.102764295	27.76529099	0.7243116396
41.75	6,363,752.36385	4,623.869266317	27.76650202	0.7286749627
42.00	6,364,030.36636	4,651.636980707	27.76771439	0.7330382858
42.25	6,364,308.64425	4,679.405908713	27.76892801	0.7374016090
42.50	6,364,587.17637	4,707.176051489	27.77014278	0.7417649321
42.75	6,364,865.94158	4,734.947410100	27.77135861	0.7461282552
43.00	6,365,144.91867	4,762.719985516	27.77257542	0.7504915784
43.25	6,365,424.08646	4,790.493778615	27.77379310	0.7548549015
43.50	6,365,703.42372	4,818.268790183	27.77501157	0.7592182246
43.75	6,365,982.90920	4,846.045020913	27.77623073	0.7635815477
44.00	6,366,262.52166	4,873.822471405	27.77745049	0.7679448709
44.25	6,366,542.23982	4,901.601142168	27.77867076	0.7723081940
44.50	6,366,822.04239	4,929.381033616	27.77989145	0.7766715171
44.75	6,367,101.90810	4,957.162146070	27.78111245	0.7810348403
45.00	6,367,381.81562	4,984.944479760	27.78233369	0.7853981634
45.25	6,367,661.74365	5,012.728034822	27.78355506	0.7897614865
45.50	6,367,941.67088	5,040.512811298	27.78477648	0.7941248097
45.75	6,368,221.57597	5,068.298809139	27.78599784	0.7984881328
46.00	6,368,501.43760	5,096.086028202	27.78721906	0.8028514559
46.25	6,368,781.23446	5,123.874468250	27.78844005	0.8072147790
46.50	6,369,060.94520	5,151.664128954	27.78966070	0.8115781022
46.75	6,369,340.54851	5,179.455009893	27.79088094	0.8159414253
47.00	6,369,620.02306	5,207.247110550	27.79210066	0.8203047484
47.25	6,369,899.34754	5,235.040430317	27.79331977	0.8246680716
47.50	6,370,178.5064	5,262.834968493	27.79453818	0.8290313947
47.75	6,370,457.46105	5,290.630724284	27.79575579	0.8333947178
48.00	6,370,736.20750	5,318.427696803	27.79697252	0.8377580410
48.25	6,371,014.71869	5,346.225885070	27.79818827	0.8421213641
48.50	6,371,292.97336	5,374.025288011	27.79940294	0.8464846872
48.75	6,371,570.95027	5,401.825904461	27.80061645	0.8508480103
49.00	6,371,848.62817	5,429.627733162	27.80182870	0.8552113335
49.25	6,372,125.98585	5,457.430772762	27.80303960	0.8595746566
49.50	6,372,403.00212	5,485.235021819	27.80424906	0.8639379797
49.75	6,372,679.65580	5,513.040478797	27.80545698	0.8683013029
50.00	6,372,955.92574	5,540.847142066	27.80666327	0.8726646260

Degree φ	M(φ)	KM up to φ	delta KM	Radian=φ
50.25	6,373,231.79080	5,568.655009908	27.80786784	0.8770279491
50.50	6,373,507.22990	5,596.464080508	27.80907060	0.8813912723
50.75	6,373,782.22196	5,624.274351963	27.81027145	0.8857545954
51.00	6,374,056.74594	5,652.085822276	27.81147031	0.8901179185
51.25	6,374,330.78082	5,679.898489359	27.81266708	0.8944812416
51.50	6,374,604.30562	5,707.712351032	27.81386167	0.8988445648
51.75	6,374,877.29941	5,735.527405023	27.81505399	0.9032078879
52.00	6,375,149.74128	5,763.343648970	27.81624395	0.9075712110
52.25	6,375,421.61034	5,791.161080420	27.81743145	0.9119345342
52.50	6,375,692.88579	5,818.979696827	27.81861641	0.9162978573
52.75	6,375,963.54682	5,846.799495556	27.81979873	0.9206611804
53.00	6,376,233.57268	5,874.620473881	27.82097832	0.9250245036
53.25	6,376,502.94269	5,902.442628985	27.82215510	0.9293878267
53.50	6,376,771.63617	5,930.265957961	27.82332898	0.9337511498
53.75	6,377,039.63252	5,958.090457813	27.82449985	0.9381144729
54.00	6,377,306.91119	5,985.916125454	27.82566764	0.9424777961
54.25	6,377,573.45166	6,013.742957708	27.82683225	0.9468411192
54.50	6,377,839.23347	6,041.570951309	27.82799360	0.9512044423
54.75	6,378,104.23622	6,069.400102902	27.82915159	0.9555677655
55.00	6,378,368.43958	6,097.230409043	27.83030614	0.9599310886
55.25	6,378,631.82324	6,125.061866201	27.83145716	0.9642944117
55.50	6,378,894.36697	6,152.894470755	27.83260455	0.9686577349
55.75	6,379,156.05061	6,180.728218995	27.83374824	0.9730210580
56.00	6,379,416.85404	6,208.563107125	27.83488813	0.9773843811
56.25	6,379,676.75723	6,236.399131261	27.83602414	0.9817477042
56.50	6,379,935.74019	6,264.236287431	27.83715617	0.9861110274
56.75	6,380,193.78300	6,292.074571576	27.83828415	0.9904743505
57.00	6,380,450.86583	6,319.913979551	27.83940797	0.9948376736
57.25	6,380,706.96889	6,347.754507124	27.84052757	0.9992009968
57.50	6,380,962.07249	6,375.596149977	27.84164285	1.0035643199
57.75	6,381,216.15699	6,403.438903705	27.84275373	1.0079276430
58.00	6,381,469.20284	6,431.282763821	27.84386012	1.0122909662
58.25	6,381,721.19056	6,459.127725749	27.84496193	1.0166542893
58.50	6,381,972.10074	6,486.973784830	27.84605908	1.0210176124
58.75	6,382,221.91407	6,514.820936320	27.84715149	1.0253809355
59.00	6,382,470.61129	6,542.669175392	27.84823907	1.0297442587
59.25	6,382,718.17325	6,570.518497133	27.84932174	1.0341075818
59.50	6,382,964.58087	6,598.368896548	27.85039942	1.0384709049
59.75	6,383,209.81517	6,626.220368560	27.85147201	1.0428342281
60.00	6,383,453.85723	6,654.072908008	27.85253945	1.0471975512
60.25	6,383,696.68824	6,681.926509647	27.85360164	1.0515608743
60.50	6,383,938.28948	6,709.781168154	27.85465851	1.0559241975

Degree φ	M(φ)	KM up to φ	delta KM	Radian=φ
60.75	6,384,178.64230	6,737.636878121	27.85570997	1.0602875206
61.00	6,384,417.72818	6,765.493634062	27.85675594	1.0646508437
61.25	6,384,655.52865	6,793.351430407	27.85779635	1.0690141668
61.50	6,384,892.02537	6,821.210261508	27.85883110	1.0733774900
61.75	6,385,127.20009	6,849.070121636	27.85986013	1.0777408131
62.00	6,385,361.03464	6,876.931004984	27.86088335	1.0821041362
62.25	6,385,593.51098	6,904.792905664	27.86190068	1.0864674594
62.50	6,385,824.61114	6,932.655817712	27.86291205	1.0908307825
62.75	6,386,054.31727	6,960.519735083	27.86391737	1.0951941056
63.00	6,386,282.61164	6,988.384651656	27.86491657	1.0995574288
63.25	6,386,509.47659	7,016.25561232	27.86590958	1.1039207519
63.50	6,386,734.89460	7,044.117457537	27.86689630	1.1082840750
63.75	6,386,958.84824	7,071.985334219	27.86787668	1.1126473981
64.00	6,387,181.32020	7,099.854184850	27.86885063	1.1170107213
64.25	6,387,402.29327	7,127.724002929	27.86981808	1.1213740444
64.50	6,387,621.75037	7,155.594781876	27.87077895	1.1257373675
64.75	6,387,839.67451	7,183.466515042	27.87173317	1.1301006907
65.00	6,388,056.04884	7,211.339195700	27.87268066	1.1344640138
65.25	6,388,270.85661	7,239.212817051	27.87362135	1.1388273369
65.50	6,388,484.08121	7,267.087372225	27.87455517	1.1431906601
65.75	6,388,695.70612	7,294.962854276	27.87548205	1.1475539832
66.00	6,388,905.71497	7,322.839256189	27.87640191	1.1519173063
66.25	6,389,114.09149	7,350.716570877	27.87731469	1.1562806294
66.50	6,389,320.81954	7,378.594791183	27.87822031	1.1606439526
66.75	6,389,525.88312	7,406.473909880	27.87911870	1.1650072757
67.00	6,389,729.26633	7,434.353919668	27.88000979	1.1693705988
67.25	6,389,930.95343	7,462.234813184	27.88089352	1.1737339220
67.50	6,390,130.92877	7,490.116582990	27.88176981	1.1780972451
67.75	6,390,329.17687	7,517.999221586	27.88263860	1.1824605682
68.00	6,390,525.68235	7,545.882721400	27.88349981	1.1868238914
68.25	6,390,720.42998	7,573.767074796	27.88435340	1.1911872145
68.50	6,390,913.40466	7,601.652274071	27.88519927	1.1955505376
68.75	6,391,104.59142	7,629.538311456	27.88603739	1.1999138607
69.00	6,391,293.97544	7,657.425179117	27.88686766	1.2042771839
69.25	6,391,481.54202	7,685.312869158	27.88769004	1.2086405070
69.50	6,391,667.27661	7,713.201373615	27.88850446	1.2130038301
69.75	6,391,851.16480	7,741.090684464	27.88931085	1.2173671533
70.00	6,392,033.19232	7,768.980793617	27.89010915	1.2217304764
70.25	6,392,213.34504	7,796.871692925	27.89089931	1.2260937995
70.50	6,392,391.60897	7,824.763374177	27.89168125	1.2304571227
70.75	6,392,567.97028	7,852.655829101	27.89245492	1.2348204458
71.00	6,392,742.41526	7,880.549049366	27.89322026	1.2391837689

Degree φ	M(φ)	KM up to φ	delta KM	Radian=φ
71.25	6,392,914.93037	7,908.443026580	27.89397721	1.2435470920
71.50	6,393,085.50222	7,936.337752293	27.89472571	1.2479104152
71.75	6,393,254.11755	7,964.233217999	27.89546571	1.2522737383
72.00	6,393,420.76326	7,992.129415130	27.89619713	1.2566370614
72.25	6,393,585.42640	8,020.026335066	27.89691994	1.2610003846
72.50	6,393,748.09418	8,047.923969126	27.89763406	1.2653637077
72.75	6,393,908.75395	8,075.822308578	27.89833945	1.2697270308
73.00	6,394,067.39323	8,103.721344633	27.89903605	1.2740903540
73.25	6,394,223.99968	8,131.621068446	27.89972381	1.2784536771
73.50	6,394,378.56112	8,159.521471123	27.90040268	1.2828170002
73.75	6,394,531.06554	8,187.422543714	27.90107259	1.2871803233
74.00	6,394,681.50108	8,215.324277217	27.90173350	1.2915436465
74.25	6,394,829.85604	8,243.226662580	27.90238536	1.2959069696
74.50	6,394,976.11887	8,271.129690699	27.90302812	1.3002702927
74.75	6,395,120.27820	8,299.033352421	27.90366172	1.3046336159
75.00	6,395,262.32281	8,326.937638543	27.90428612	1.3089969390
75.25	6,395,402.24164	8,354.842539814	27.90490127	1.3133602621
75.50	6,395,540.02382	8,382.748046935	27.90550712	1.3177235853
75.75	6,395,675.65861	8,410.654150559	27.90610362	1.3220869084
76.00	6,395,809.13546	8,438.560841293	27.90669073	1.3264502315
76.25	6,395,940.44398	8,466.468109700	27.90726841	1.3308135546
76.50	6,396,069.57394	8,494.375946295	27.90783660	1.3351768778
76.75	6,396,196.51529	8,522.284341551	27.90839526	1.3395402009
77.00	6,396,321.25815	8,550.193285897	27.90894435	1.3439035240
77.25	6,396,443.79280	8,578.102769718	27.90948382	1.3482668472
77.50	6,396,564.10969	8,606.012783360	27.91001364	1.3526301703
77.75	6,396,682.19946	8,633.923317124	27.91053376	1.3569934934
78.00	6,396,798.05290	8,661.834361273	27.91104415	1.3613568166
78.25	6,396,911.66099	8,689.745906029	27.91154476	1.3657201397
78.50	6,397,023.01488	8,717.657941577	27.91203555	1.3700834628
78.75	6,397,132.10589	8,745.570458060	27.91251648	1.3744467859
79.00	6,397,238.92553	8,773.483445588	27.91298753	1.3788101091
79.25	6,397,343.46545	8,801.396894230	27.91344864	1.3831734322
79.50	6,397,445.71753	8,829.310794023	27.91389979	1.3875367553
79.75	6,397,545.67378	8,857.225134966	27.91434094	1.3919000785
80.00	6,397,643.32642	8,885.139907025	27.91477206	1.3962634016
80.25	6,397,738.66783	8,913.055100131	27.91519311	1.4006267247
80.50	6,397,831.69058	8,940.970704184	27.91560405	1.4049900479
80.75	6,397,922.38741	8,968.886709051	27.91600487	1.4093533710
81.00	6,398,010.75127	8,996.803104568	27.91639552	1.4137166941
81.25	6,398,096.77524	9,024.719880540	27.91677597	1.4180800172
81.50	6,398,180.45263	9,052.637026743	27.91714620	1.4224433404

Degree φ	M(φ)	KM up to φ	delta KM	Radian=φ
81.75	6,398,261.77691	9,080.554532924	27.91750618	1.4268066635
82.00	6,398,340.74173	9,108.472388801	27.91785588	1.4311699866
82.25	6,398,417.34095	9,136.390584067	27.91819527	1.4355333098
82.50	6,398,491.56857	9,164.309108385	27.91852432	1.4398966329
82.75	6,398,563.41881	9,192.227951396	27.91884301	1.4442599560
83.00	6,398,632.88607	9,220.147102714	27.91915132	1.4486232792
83.25	6,398,699.96493	9,248.066551930	27.91944922	1.4529866023
83.50	6,398,764.65015	9,275.986288610	27.91973668	1.4573499254
83.75	6,398,826.93668	9,303.906302299	27.92001369	1.4617132485
84.00	6,398,886.81967	9,331.826582521	27.92028022	1.4660765717
84.25	6,398,944.29444	9,359.747118778	27.92053626	1.4704398948
84.50	6,398,999.35651	9,387.667900553	27.92078177	1.4748032179
84.75	6,399,052.00158	9,415.588917307	27.92101675	1.4791665411
85.00	6,399,102.22554	9,443.510158487	27.92124118	1.4835298642
85.25	6,399,150.02446	9,471.431613520	27.92145503	1.4878931873
85.50	6,399,195.39464	9,499.353271817	27.92165830	1.4922565105
85.75	6,399,238.33250	9,527.275122771	27.92185095	1.4966198336
86.00	6,399,278.83472	9,555.197155764	27.92203299	1.509831567
86.25	6,399,316.89812	9,583.119360160	27.92220440	1.5053464798
86.50	6,399,352.51974	9,611.041725312	27.92236515	1.5097098030
86.75	6,399,385.69680	9,638.964240560	27.92251525	1.5140731261
87.00	6,399,416.42669	9,666.886895230	27.92265467	1.5184364492
87.25	6,399,444.70703	9,694.809678641	27.92278341	1.5227997724
87.50	6,399,470.53561	9,722.732580100	27.92290146	1.5271630955
87.75	6,399,493.91041	9,750.655588904	27.92300880	1.5315264186
88.00	6,399,514.82961	9,778.578694342	27.92310544	1.5358897418
88.25	6,399,533.29157	9,806.501885696	27.92319135	1.5402530649
88.50	6,399,549.29485	9,834.425152242	27.92326655	1.5446163880
88.75	6,399,562.83820	9,862.348483249	27.92333101	1.5489797111
89.00	6,399,573.92057	9,890.271867981	27.92338473	1.5533430343
89.25	6,399,582.54108	9,918.195295697	27.92342772	1.5577063574
89.50	6,399,588.69908	9,946.118755656	27.92345996	1.5620696805
89.75	6,399,592.39406	9,974.042237110	27.92348145	1.5664330037
90.00	6,399,593.62576	10,001.965729313	27.92349220	1.5707963268

The most accurate value taken in Weintrit, [36], is 10,001,965.72931270 m which is consistent with the above 10,001,965.72931260 m to the micrometer (10^{-6}). This implies that our Trapezoid-rule calculations in meters on a meridian is valid and sufficiently accurate for GIS use. This is also consistent with the Wikipedia "Latitude" entry which cites 10,001.965729 km (see [32]). The Simpson rule method is described in [37]. Simpson rule summation is a bit more accurate, but the Trapezoid rule works sufficiently for the needs of GIS accuracy using quarter degree latitude steps.

The accuracy for east-west parallel for longitude does not need the numeric integration, and so it simply trigonometric functions is probably more accurate than along meridians.

C.3 Area of a Surface

The area integral is like the length integrals, except instead of using Pythagorean summation, it multiplies horizontal λ-distances and vertical to get area in square meters. Essentially, the area locally for a ΔΦ, ΔΛ rectangle (the minimum bounding rectangle) which is divided into horizontal stripes and then intersected with the area geometry. This gives a set of sub-strips where the boundary of the area crosses the stripe. This gives us a set of horizontal sections of lengths "$\Delta\lambda_{1,j}, \Delta\lambda_{2,j}, \ldots, \Delta\lambda_{n_i-1,j}, \Delta\lambda_{n_i,j}$" for the latitude division lines $\varphi_0, \varphi_1, \ldots, \varphi_m;\ 0, \ldots, j, \ldots m$ with $\Delta\varphi_j = \varphi_j - \varphi_{j-1}$ and a single vertical height for each stripe $\Delta\varphi_j$.

For each sub-stripe, the area integral is the product of the corresponding meter distance which is exactly what you do with length, but this time you multiply to get local areas and then add the little strips into a total area. An example is in Table 3.

Eq 108.

$$A_S = \iint_W \sqrt{EG}\, d\varphi d\lambda$$

$$E(\varphi) = \left(\mathrm{N}'(\varphi)\cos\varphi + \mathrm{N}(\varphi)\sin\varphi\right)^2 + \frac{b^4}{a^4}\left(\mathrm{N}'(\varphi)\sin\varphi + \mathrm{N}(\varphi)\cos\varphi\right)^2$$

$$= M^2(\varphi) = \left[a(1-e^2)(1-e^2\sin^2\varphi)^{-3/2}\right]^2$$

$$\mathrm{G}(\varphi) = \mathrm{N}(\varphi)^2\cos^2\varphi = \rho^2(\varphi)$$

$$\sqrt{E(\varphi)} = M(\varphi) = a(1-e^2)(1-e^2\sin^2\varphi)^{-3/2}$$

$$\sqrt{G(\varphi)} = \rho(\varphi) = \mathrm{N}(\varphi)\cos\varphi$$

$$A_S = \sum\left[\left(M(\varphi_i) + M(\varphi_{i-1})\right)\Delta\varphi_i\right]\left[\left(\rho(\varphi_i) + \rho(\varphi_{i-1})\right)\Delta\lambda_i\right]$$

For each stripe "j" of height $\Delta\varphi_j$ there are

sub-stips of lengths $\Delta\lambda_{1,j}, \Delta\lambda_{2,j}, \ldots, \Delta\lambda_{n_i-1,j}, \Delta\lambda_{n_i,j}$

Eq 109.

$$\Delta\lambda_j = \sum_{i=1}^{n_i}\Delta\lambda_{i,j} \text{ with area = [height]}\times\text{[length]:}$$

$$A \cong \left[\sum_{j=1}^{m}\left(\frac{M(\varphi_j) + M(\varphi_{j-1})}{2}\right)\Delta\varphi_j\right]\left[\left(\frac{\rho(\varphi_i) + \rho(\varphi_{i-1})}{2}\right)(\Delta\lambda_j)\right]$$

In the numeric approximation for the area integral is a double summation, both in the latitude and the longitude directions. The table below expresses angles as degrees, but the units of latitude and longitude are in radians in the integrals.

$$\varphi_{i=0...8} = \{0, 0.125, 0.250, 0.375, 0.5, 0.625, 0.750, 0.875, 1.00\}; \Delta\varphi_i = \varphi_i - \varphi_{i-1}$$

$$\lambda_{j=0...8} = \{0, 0.125, 0.250, 0.375, 0.5, 0.625, 0.750, 0.875.1.00\}; \Delta\lambda_i = \lambda_i - \lambda_{i-1} = 0.125^{\circ}$$

$$\Delta\lambda = \sum_{j=1}^{8} \Delta\lambda_j = 1^{\circ} = \pi / 180 = 0.0174532925199432957692369 \text{ radians}$$

Eq 110.

$$A \cong \sum_{i=1}^{8} \left(\frac{M(\varphi_i) + M(\varphi_{i-1})}{2} \Delta\varphi_i \right) \left[\sum_{j=1}^{8} \left(\frac{\rho(\varphi_j) + \rho(\varphi_{j-1})}{2} \Delta\lambda_j \right) \right]$$

$$\cong \left[\sum_{i=1}^{8} \left(\frac{M(\varphi_i) + M(\varphi_{i-1})}{2} \Delta\varphi_i \right) \left(\frac{\rho(\varphi_i) + \rho(\varphi_{i-1})}{2} \Delta\lambda \right) \right]$$

$$\left(\frac{M(\varphi_i) + M(\varphi_{i-1})}{2} \Delta\varphi_i \right) = \text{Height of strip between } \varphi_i \text{ and } \varphi_{i-1}.\ \Delta\varphi_i = \left(1/8\right)^{\circ} \text{ as radians}$$

$$\left(\frac{\rho(\varphi_i) + \rho(\varphi_{i-1})}{2} \Delta\lambda \right) = \text{Length of strip between } \varphi_i \text{ and } \varphi_{i-1}.\ \Delta\lambda = 1^{\circ} \text{ as radians}$$

The corresponding angle in radians are 0.00000, 0.002181, 0.004363, 0.006544, .008726, 0.010908, 0.013089, 0.015271, 0.017453.

The final value for the area of a degree square in latitude and longitude (along the equator) is 12,308.814 square kilometers. This is slightly different from the information on lengths of degrees of latitude and longitude. A degree of longitude (λ) at latitude φ=0° is 111.319490 km and at latitude φ=1° is 111.31838 km for an average width is 111.31893, which implies that on a flat surface, the trapezoid would have an area of 12,309.023 which is about 0.2 square kilometers too large. In a plane the average is based on a linear growth, but on the ellipsoid the average is mainly associated to the cos φ which is near flat near φ=0 which goes linear near 90° (i.e. near the pole). So, the actual area (as latitude increases down the columns below the rapidity of change increases as the squares moves nearer the pole) is slightly larger than a planar approximation.

Table 3. Area of 1˚Latitude by 1˚Longitude at the Equator via Squares (sq. meters)

Lat↓	Long → .000	.125	.250	.375	.500	.625	.750	.875	1.00
.000	192,329,729.221821	192,329,729.221821	192,329,729.221821	192,329,729.221821	192,329,729.221821	192,329,729.221821	192,329,729.221821	192,329,729.221821	1,538,637,833.77457
.125	192,330,236.673325	192,330,236.673325	192,330,236.673325	192,330,236.673325	192,330,236.673325	192,330,236.673325	192,330,236.673325	192,330,236.673325	1,538,641,893.38660
.250	192,329,847.085907	192,329,847.085907	192,329,847.085907	192,329,847.085907	192,329,847.085907	192,329,847.085907	192,329,847.085907	192,329,847.085907	1,538,638,776.68726
.375	192,328,560.440212	192,328,560.440212	192,328,560.440212	192,328,560.440212	192,328,560.440212	192,328,560.440212	192,328,560.440212	192,328,560.440212	1,538,628,483.52170
.500	192,326,376.720375	192,326,376.720375	192,326,376.720375	192,326,376.720375	192,326,376.720375	192,326,376.720375	192,326,376.720375	192,326,376.720375	1,538,611,013.76300
.625	192,323,295.914023	192,323,295.914023	192,323,295.914023	192,323,295.914023	192,323,295.914023	192,323,295.914023	192,323,295.914023	192,323,295.914023	1,538,586,367.31218
.750	192,319,318.012272	192,319,318.012272	192,319,318.012272	192,319,318.012272	192,319,318.012272	192,319,318.012272	192,319,318.012272	192,319,318.012272	1,538,554,544.09818
.875	192,314,443.009734	192,314,443.009734	192,314,443.009734	192,314,443.009734	192,314,443.009734	192,314,443.009734	192,314,443.009734	192,314,443.009734	1,538,515,544.07788
1.00									
Sum	1,538,601,807.07767	1,538,601,807.07767	1,538,601,807.07767	1,538,601,807.07767	1,538,601,807.07767	1,538,601,807.07767	1,538,601,807.07767	1,538,601,807.07767	12,308,814,456.6214

$$\text{Each square } = \left(\frac{\rho(\varphi_i)+\rho(\varphi_{i-1})}{2}\right)\left(\frac{M(\varphi_i)+M(\varphi_{i-1})}{2}\right)(\Delta\lambda)(\Delta\varphi)$$

Table 4. Area of 1° Latitude by 1° Longitude at the Equator via Stripes (sq. meters)

Latitude in radians	Average M(ρ)	Average ρ(φ)	Stripe Height	Stripe Length	Area of Row
0.0000000	6,335,439.32729282	6,378,137.0000000	13,821.78480800	111,352.96452167	1,538,637,833.77457
0.00218166	6,335,439.63009043	6,378,168.39746324	13,821.78612920	111,351.78018338	1,538,641,893.38660
0.00436332	6,335,440.53847765	6,378,169.43746257	13,821.78877157	111,348.59284744	1,538,638,776.68726
0.00654498	6,335,442.05243760	6,378,140.11932799	13,821.79273506	111,342.41803770	1,538,628,483.52170
0.00872665	6,335,444.17194220	6,378,080.44253395	13,821.79801961	111,332.26756270	1,538,611,013.76300
0.01090831	6,335,446.89695210	6,377,990.40669941	13,821.80462511	111,317.15280954	1,538,586,367.31218
0.01308997	6,335,450.22741673	6,377,870.01158780	13,821.81255144	111,296.10221165	1,538,554,544.09818
0.01527163	6,335,454.16327427	6,377,719.25710712	13,821.82179845	110,943.09955475	1,538,515,544.07787
0.01745329					
				12,308,814,456.6214	

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